Solutions - 6.4

(4) Let l_2 be the space of sequences $x = \{x_1, x_2, \ldots\}$ such that $\sum_{i=1}^{\infty} x_i^2$ converges. Define $e_n = \{0, 0, \ldots, 0, 1, 0, \ldots\}$ where all terms are 0 except the nth term is 1. Define

$$d(x,y) = \left(\sum_{i=1}^{\infty}(x_i - y_i)^2\right)^{\frac{1}{2}}.$$

Show that $\{e_n\}$ is bounded but not compact.

First, we'll check bounded.

$$d(e_n, 0) = \left(\sum_{i=1}^{\infty}(e_n - 0)^2\right)^{\frac{1}{2}} = \left(0^2 + 0^2 + \cdots + 0^2 + 1^2 + 0^2 + \cdots\right)^{\frac{1}{2}} = 1$$

so all e_n are contained in the ball $B(0,2)$ (I used 2, but any radius bigger than 1 would work). So the set is bounded.

Now compact. Suppose e_{n_k} is any subsequence. Then

$$d(e_{n_i}, e_{n_j}) = \left(\cdots + 0^2 + 1^2 + 0^2 + \cdots + 0^2 + 1^2 + 0^2 + \cdots\right)^{\frac{1}{2}} = \sqrt{2}$$

The distance formula has a $(1-0)^2$ in the i spot and a $(0-1)^2$ in the j spot (assuming $i \neq j$). Since all terms in any subsequence are a distance $\sqrt{2}$ from each other, no subsequence could possibly converge. Therefore the set is not compact.