**Solutions - 9.1 continued**

(5) Show whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n+1}{n^3}$$

Notice that

$$\frac{n+1}{n^3} = \frac{1}{n^2} + \frac{1}{n^3}$$

Since $\frac{1}{n^2}$ and $\frac{1}{n^3}$ both converges (P-Series), their sum also converges.

(9) Show whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n}{e^n}$$

Use the integral test. First note that by l'Hopital's rule,

$$\lim_{n\to\infty} \frac{n}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = 0$$

Then

$$\begin{aligned} \int_1^{\infty} \frac{x}{e^x} dx &= \int_1^{\infty} xe^{-x} dx \\ &= \lim_{t\to\infty} -e^{-x} - xe^{-x}\Big|_1^t \\ &= 0 - (-e^{-1} - e^{-1}) \\ &= 2e^{-1} \end{aligned}$$

(We use integration by parts to get the antiderivative.)Finally, note the since the derivative $(xe^{-x})' = e^{-x}(1-x)$ is negative for $x > 1$, the function is non-increasing. All the requirements for the integral test apply, and the integral converges, so the series converges.

(13) For what values of $p$ does the series

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$$

converge? Justify your answer.

The series converges for $p > 1$. There are several ways to check this, I'll do two here.

The first is to compare to a P-series. $\ln n$ grows "slower" than $n^\epsilon$ for any $\epsilon > 0$, since

$$\lim_{n\to\infty} \frac{\ln n}{n^\epsilon} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\epsilon n^{\epsilon-1}} = \lim_{n\to\infty} \frac{1}{\epsilon n^\epsilon} = 0$$

Since the limit is 0, we can find $N$ such that $n > N$ implies $\frac{\ln n}{n^\epsilon} < 1$. Consequently, $\frac{\ln n}{n^p} < \frac{1}{n^{p-\epsilon}}$. So the series converges (by comparison to p-series) if $p - \epsilon > 1$. Since $\epsilon$ can be arbitrarily small, we have convergence for $p > 1$. Although this argument only applies to terms after $N$, there are finite terms before that, so we still have convergence when considering all $n$.

On the other side, $\ln n > 1$ for $n > e$. Then $\frac{\ln n}{n^p} > \frac{1}{n^p}$ so the given series diverges (by comparison to p-series) for $p \geq 1$.

For a second method, try integral test with substitution. Note that the limit of the terms is 0 (need to use l'Hopital's once), and that the derivative $\frac{n^{p-1} - p \ln n n^{p-1}}{n^{2p}} = \frac{1-p\ln n}{n^{p+1}}$ is negative for $n > e^{\frac{1}{p}}$. So the integral test applies, at least for terms past the point where the derivative goes negative. To do the integral, we substitute $u = \ln x$ (so $x = e^u$).

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{x^p} dx &= \int_0^\infty \frac{u}{x^p} x du \\
&= \int_0^\infty \frac{u}{x^{p-1}} du \\
&= \int_0^\infty \frac{u}{e^{(p-1)u}} du \\
&= \int_0^\infty u e^{(1-p)u} du \\
&= \left( \frac{-1}{p-1} e^{(1-p)u} + \frac{u}{1-p} e^{(1-p)u} \right|_0^\infty
\end{aligned}$$

The integral converges as long as $e$ has a negative power; that is, if $p > 1$. If $p < 1$, the integral diverges. If $p = 1$, the integrand is just $u$ and the integral diverges.