Solutions - 9.1

(3) Show whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$$

$\frac{1}{n(n+2)} < \frac{1}{n^2}$ for all positive integers n, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges since it is a p-series with $p = 2 > 1$. By comparison, the given series converges as well.

(4) Show whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n2^n}$$

$\frac{1}{n2^n} \leq \frac{1}{2^n}$ for all positive integers n, and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges since it is geometric with $r = \frac{1}{2}$. By comparison, the given series converges.

(11) Prove Theorem 9.4a. Suppose $u_n \geq 0$ for all $n \in \mathbb{N}$. If $u_n \leq a_n$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} an$ converges, then $\sum_{n=1}^{\infty} u_n$ converges and $\sum_{n=1}^{\infty} u_n \leq \sum_{n=1}^{\infty} a_n$.

Since $u_n \leq a_n$ for all $n \in \mathbb{N}$, $\sum_{n=1}^{k} u_n \leq \sum_{n=1}^{k} a_n$ for all $k \in \mathbb{N}$. Note the sequence $s_k = \sum_{n=1}^{k} u_n$ is non-decreasing since $u_n \geq 0$. $\sum_{n=1}^{k} a_n$ is also non-decreasing so $\sum_{n=1}^{k} a_n \leq \sum_{n=1}^{\infty} a_n$ for all $k \in \mathbb{N}$, so $s_k = \sum_{n=1}^{k} u_n$ is bounded above. By Axiom C, $\lim_{k\to\infty} s_k = \sum_{n=1}^{\infty} u_n$ converges.