Solutions - 9.2 continued

(1) Show whether the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}10^n}{n!}$$

Use Ratio Test.

$$\begin{aligned}\lim_{n\to\infty} \frac{\left|\frac{(-1)^n 10^{n+1}}{(n+1)!}\right|}{\left|\frac{(-1)^{n-1}10^n}{n!}\right|} &= \lim_{n\to\infty} \frac{10}{n+1} \\ &= 0 < 1\end{aligned}$$

The series is absolutely convergent by the Ratio Test.

(2) Show whether the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n!}{10^n}$$

Use Ratio Test.

$$\begin{aligned}\lim_{n\to\infty} \frac{\left|\frac{(-1)^n (n+1)!}{10^{n+1}}\right|}{\left|\frac{(-1)^{n-1}n!}{10^n}\right|} &= \lim_{n\to\infty} \frac{n+1}{10} \\ &= +\infty\end{aligned}$$

The series is divergent by the Ratio Test.

(5) Show whether the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n (4/3)^n}{n^4}$$

Use Ratio Test.

$$\begin{aligned}\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1} (4/3)^{n+1}}{(n+1)^4}\right|}{\left|\frac{(-1)^n (4/3)^n}{n^4}\right|} &= \lim_{n\to\infty} \frac{4n^4}{3(n+1)^4} \\ &= \lim_{n\to\infty} \frac{4}{3(1+\frac{1}{n})^4} \\ &= \frac{4}{3} > 1\end{aligned}$$

The series is divergent by the Ratio Test.

(16) Show whether the following series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty}\left(\frac{n}{2n+1}\right)^n$$

Use Root Test.

$$\begin{aligned}\lim_{n\to\infty}\left|\left(\frac{n}{2n+1}\right)^n\right|^{\frac{1}{n}} &= \lim_{n\to\infty}\left|\frac{n}{2n+1}\right| \\ &= \lim_{n\to\infty}\frac{1}{2} \qquad \text{by l'Hopital's} \\ &= \frac{1}{2}<1\end{aligned}$$

The series is absolutely convergent by the Ratio Test.

(17) Find all values of x for which the power series converges.

$$\sum_{n=1}^{\infty}(n+1)x^n$$

Use Ratio Test.

$$\begin{aligned}\lim_{n\to\infty}\frac{|(n+2)x^{n+1}|}{|(n+1)x^n|} &= \lim_{n\to\infty}\frac{(n+2)|x|}{n+1} \\ &= \lim_{n\to\infty}|x|\frac{1+\frac{2}{n}}{1+\frac{1}{n}} \\ &= |x|\end{aligned}$$

The series is absolutely convergent for $|x|<1$ and divergent for $|x|>1$. If $|x|=1$, then $u_n=\pm(n+1)$ and $\lim_{n\to\infty}(n+1)x^n=\infty$ so the series diverges.

(20) Find all values of x for which the power series converges.

$$\sum_{n=1}^{\infty}\frac{n(x+2)^n}{2^n}$$

Use Ratio Test.

$$\begin{aligned} \lim_{n\to\infty} \frac{\left|\frac{(n+1)(x+2)^{n+1}}{2^{n+1}}\right|}{\left|\frac{n(x+2)^n}{2^n}\right|} &= \lim_{n\to\infty} \frac{(n+1)|x+2|}{2n} \\ &= \lim_{n\to\infty} \frac{|x+2|}{2} \frac{1+\frac{1}{n}}{1} \\ &= \frac{|x+2|}{2} \end{aligned}$$

The series is absolutely convergent for $|x+2| < 2$ and divergent for $|x+2| > 2$. If $|x+2| = 2$, then $u_n = \pm n$ and $\frac{n(x+2)^n}{2^n} = \infty$ so the series diverges.

(29a) Find c_n so that the series

$$\sum_{n=1}^{\infty} c_n x^n$$

only converges for $x = 0$.

If $c_n = n!$, then

$$\frac{(n+1)!x^{n+1}}{n!x^n} = (n+1)x$$

These ratios go to ∞ unless $x = 0$, so the series diverges by Ratio Test unless $x = 0$.

(29b) Find c_n so that the series

$$\sum_{n=1}^{\infty} c_n x^n$$

converges for all values of x.

If $c_n = \frac{1}{n!}$, then

$$\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \frac{x}{n+1}$$

These ratios go to 0 for any x, so the series converges by Ratio Test for any x.