Solutions - 9.3

(1) Show that the sequence $\{f_n\}$ converges to f for each x in I. Determine whether or not the convergence is uniform.

$$f_n(x) = \frac{2x}{1+nx}, \qquad f(x) = 0, \qquad I = [0,1]$$

If $x = 0$, each function is 0 so convergence is clear. If $x > 0$, then the denominators go to ∞ so the functions go to 0.

Let $\epsilon > 0$ be given. Assume $n \geq N$. If $x = 0$, any N works. Assume $x > 0$.

$$\begin{aligned} \left|\frac{2x}{1+nx} - 0\right| &= \frac{2x}{1+nx} \\ &< \frac{2x}{nx} \\ &= \frac{2}{n} \\ &< \epsilon \end{aligned}$$

The last step holds if $N > \frac{2}{\epsilon}$. Since N does not depend on x, convergence is uniform.

(4) Show that the sequence $\{f_n\}$ converges to f for each x in I. Determine whether or not the convergence is uniform.

$$f_n(x) = \frac{n^3x}{1+n^4x^2}, \qquad f(x) = 0, \qquad I = [1,\infty)$$

Convergence follows from l'Hopital's rule.

$$\lim n \to \infty \frac{n^3x}{1+n^4x^2} = \lim n \to \infty \frac{3n^2x}{4n^3x^2} = \lim n \to \infty \frac{3}{4nx} = 0$$

Let $\epsilon > 0$ be given. Assume $n \geq N$.

$$\begin{aligned} \left|\frac{n^3x}{1+n^4x^2} - 0\right| &= \frac{n^3x}{1+n^4x^2} \\ &< \frac{n^3x}{n^4x^2} \\ &= \frac{1}{nx} \\ &\leq \frac{1}{n} \qquad \text{since } x \geq 1 \\ &< \epsilon \end{aligned}$$

The last step holds if $N > \frac{1}{\epsilon}$. Since N does not depend on x, convergence is uniform.

(5) Show that the sequence $\{f_n\}$ converges to f for each x in I. Determine whether or not the convergence is uniform.

$$f_n(x) = \frac{nx^2}{1+nx}, \qquad f(x) = x, \qquad I = [0,1]$$

Convergence follows from l'Hopital's rule (unless $x = 0$, in which case convergence is clear).

$$\lim n \to \infty \frac{nx^2}{1+nx} = \lim n \to \infty \frac{x^2}{x} = x$$

Let $\epsilon > 0$ be given. Assume $n \geq N$. If $x = 0$, any N works. Assume $x > 0$.

$$\begin{aligned} \left| \frac{nx^2}{1+nx} - x \right| &= \left| \frac{nx^2 - x - nx^2}{1+nx} \right| \\ &= \frac{x}{1+nx} \\ &< \frac{x}{nx} \\ &= \frac{1}{n} \\ &< \epsilon \end{aligned}$$

The last step holds if $N > \frac{1}{\epsilon}$. Since N does not depend on x, convergence is uniform.