The Problem Solving Competition - Problem #2 Solution

Considering the set of points

$$A = \{(i, j), 0 \leq i, j \leq 40, \text{with } i, j \text{ integers}\}$$

How many squares can be formed so that all corners of all squares belong to A, with sides parallel to the x and y axes? In other words, how many squares there with sides parallel to the axes and corners with integers coordinates between 0 and 40 inclusive?

There is 1 40 by 40 square.

There are 4 39 by 39 squares, with bottom left corners at $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$.

There are 9 38 by 38 squares, with bottom left corners at $(0, 0)$, $(1, 0)$, $(2, 0)$, $(0, 1)$, $(1, 1)$, $(2, 1)$, $(0, 2)$, $(1, 2)$ and $(2, 2)$.

There are 16 37 by 37 squares, with bottom left corners at the 16 points in a square near the origin.

This pattern of $1^2, 2^2, 3^2, 4^2$ will continue all the way up to 40, as there are 40^2 1 by 1 squares.

The total number of squares is $1^2 + 2^2 + 3^2 + 4^2 + \cdots + 40^2$. We can use the formula for sum of squares:

$$\sum_{i=1}^{n} \frac{n(n+1)(2n+1)}{6}$$

So the total number of squares is $\frac{40(41)(81)}{6} = 22140$.