The Problem Solving Competition - Problem #4 Solution

A common algebra mistake take the form $(a+b)^2 = a^2 + b^2$.

For example, claiming $(2+3)^2 = 2^2 + 3^2$ is clearly a mistake.

Note, however, that $(2+3)^2 = (2+1)^2 + (3+1)^2$.

Determine all values for m and n such that $(m+n)^2 = (m+1)^2 + (n+1)^2$ where m and n are positive integers.

Start by rewriting the equation.

$$\begin{aligned}
(m+n)^2 &= (m+1)^2 + (n+1)^2 \\
m^2 + 2mn + n^2 &= m^2 + 2m + 1 + n^2 + 2n + 1 \\
2mn &= 2m + 2n + 2 \\
mn - m &= n + 1 \\
m &= \frac{n+1}{n-1} \\
m &= 1 + \frac{2}{n-1}
\end{aligned}$$

If n is an integer, then $\frac{2}{n-1}$ is only an integer if $n = -1, 0, 2, 3$. Since n (and m) are required to be positive integers, the only solutions are $n = 2, m = 3$ and $n = 3, m = 2$.