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Sunggeum Hong · Paul Taylor · Chan Woo Yang

Weak type estimates for maximal operators with a cylindric distance function

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Abstract We establish sharp weak-type estimates for the maximal operators T_*^λ associated with cylindric Riesz means for functions on $H^p(\mathbb{R}^3)$ when $4/5 < p < 1$ and $\lambda = 3/p - 5/2$, and when $p = 4/5$ and $\lambda > 3/p - 5/2$.

Keywords Cylindric distance · Bochner-Riesz mean · Multipliers · Weak type estimates

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1 Introduction

For a Schwartz function $f \in \mathcal{S}(\mathbb{R}^3)$ let $\widehat{f}(\xi) = \int_{\mathbb{R}^3} f(y)\, e^{-i<y,\xi>}\, dy$ denote by the Fourier transform of f. We define a distance function ρ as $\rho(\xi', \xi_3) = \max\{|\xi'|, |\xi_3|\}$ with $\xi' = (\xi_1, \xi_2)$. We call this distance function ρ a cylindric distance function because its unit ball is a bounded cylinder. We are interested in

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S. Hong
Chosun University, Department of Mathematics, 501-759, Gwangju, Korea
E-mail: skhong@mail.chosun.ac.kr

P. Taylor
McMaster University, Department of Mathematics and Statistics, 1280 Main Street West, Hamilton, Ontario, L8S 4K1, Canada
E-mail: taylorp@math.mcmaster.ca

C.W. Yang (✉)
Korea University, Department of Mathematics, Seoul, 136-701, Korea
E-mail: cw_yang@korea.ac.kr

Bochner-Riesz means associated with this rough distance function ρ. More precisely we consider

$$\widehat{T^\lambda f}(\xi) = \big(1-\rho(\xi)\big)_+^\lambda \widehat{f}(\xi), \quad \xi = (\xi', \xi_3) \in \mathbb{R}^2 \times \mathbb{R}.$$

In this paper we shall actually consider a family T_ϵ^λ of convolution operators defined by

$$\widehat{T_\epsilon^\lambda f}(\xi) = \big(1-\frac{\rho(\xi)}{\epsilon}\big)_+^\lambda \widehat{f}(\xi)$$

and maximal operators T_*^λ defined by

$$T_*^\lambda f(x,t) = \sup_{\epsilon>0} |T_\epsilon^\lambda f(x,t)|, \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}.$$

The distance function ρ is not smooth along the cone $\{(\xi', \xi_3) : |\xi'| = |\xi_3|\}$. When $|\xi'| > |\xi_3| \rho(\xi', \xi_3) = |\xi'|$, and when $|\xi'| < |\xi_3| \rho(\xi', \xi_3) = |\xi_3|$. This observation leads us to guess that the multiplier T^λ may share properties with the spherical Bochner-Riesz mean and the cone multiplier. The second author rigorously justified this when $p > 1$ in [10].

This type of multipliers were previously studied by H. Luer and P. Oswald in [2] and [4,5], respectively. H. Luer considered the case where $\rho(\xi', \xi_{d+1}) = \max\{|\xi'|, |\xi_{d+1}|\}$ with $\xi' \in \mathbb{R}^d$ and $p > 1$. He used the asymptotic behavior of Bessel function to establish the kernel of T^λ is in L^1 when $d = 2$ and $\lambda > 1/2$. Interestingly he found that if $d \geq 3$ and $p > 1$, there is a restriction on p for which L^p boundedness of T^λ holds. More precisely he proved that if T^λ is bounded on $L^p(\mathbb{R}^{d+1})$ with $d \geq 3$, then $|\frac{1}{p} - \frac{1}{2}| < \frac{3}{2d}$.

This type of restriction on p was also observed by P. Oswald in [4,5] when $p < 1$. He considered a different distance function ρ_1 defined by $\rho_1(\xi) = \max\{|\xi_1|, \ldots, |\xi_n|\}$ where $\xi \in \mathbb{R}^n$. We note that this distance function coincides with cylindric distance function only when $n = 2$. He obtained sharp weak type estimates for the maximal operators when $(n-1)/n < p < 1$ with critical index $\lambda_{cube}(p) = n(1/p - 1)$, or $p = (n-1)/n$ and $\lambda > \lambda_{cube}(p)$. In his results we also have the interesting restriction on the range of p.

This type of restriction does not occur when the distance function is a smooth function such as the usual Euclidean distance function $\rho_2(\xi) = (\sum_{i=1}^n \xi_i^2)^{1/2}$. Stein, Taibleson and Weiss [9] proved that the maximal operator associated with Bochner-Riesz means of the critical index $\lambda_{B-R}(p) = n(1/p - 1/2) - 1/2$ is of weak type (p, p) for the functions in $H^p(\mathbb{R}^n)$, $p < 1$. However, for the exceptional case $p = 1$ with $\lambda_{B-R}(1) = (n-1)/2$, E. M. Stein [6] proved that there exists an $f \in H^1(\mathbb{R}^n)$ where a.e. convergence of the Bochner-Riesz means fails. For the cone multipliers, one can find a sharp estimate on $H^p(\mathbb{R}^{d+1})$, $p < 1$ in [1].

The purposes of this paper are to mathematically explain the restriction on the range of p, to find the critical index λ_p and to obtain the sharp weak type estimates for the maximal operator T_*^λ. Intuitively if the distance function ρ is smooth we can use smooth cut-off function to split the multiplier $(1-\rho(\xi))_+^\lambda$ into two multipliers: the one is supported in $\{\xi \mid \rho(\xi) \leq 1 - \epsilon_0/2\}$ and the other is supported in $\{\xi \mid \rho(\xi) \geq 1-\epsilon_0\}$ with ϵ_0 small. The first part of the multiplier is compactly supported and smooth so the operator associated with this part behaves nicely. Therefore we can only consider the second part of the multiplier. This multiplier has size ϵ_0^λ so as λ increases, the operator associated with this part has better mapping properties. However if the distance function is not smooth, the first part is not a good multiplier any more and since $1-\rho(\xi)$ is bounded below, λ does not play an important role to improve the mapping property of the associated operator. This explains the restriction on the range of p.

We shall show that the critical index λ_p of our operator T_*^λ is equal to $3/p-5/2$ when $p<1$. This is not consistent with the critical index $\lambda_p = 2|1/p-1/2|-1/2$ when $p>1$ obtained by P. Taylor in [10]. The reason why we have inconsistent critical indices when $p>1$ and when $p<1$ is that the one dimensional Bochner-Riesz multiplier plays a role when $p<1$. Actually the critical index for one dimensional Bochner-Riesz mean is 0 when $p>1$ and $1/p-1$ when $p<1$. Since the unit ball of the cylindric distance is $[-1,1]\times B_2(1,0)$ where $B_2(1,0)$ is the unit ball in 2-dimensional Euclidean space, it is likely that one dimensional Bochner-Riesz mean is involved at some point. Interestingly in this case the critical index of cylindric multiplier is the summation of those of 1- and 2-dimensional Bochner-Riesz means, that is, $\lambda_p = 0+2|1/p-1/2|-1/2$ when $p \geq 1$ and $\lambda_p = (1/p-1)+(2/p-3/2) = 3/p-5/2$ when $p<1$. This idea is also realized in the results by P. Oswald. The unit ball of $\rho_1(\xi) = \max\{|\xi_1|, ..., |\xi_n|\}$ is a cube which is also a product type domain. As is mentioned above the critical index of this case ($p<1$) is $n(1/p-1)$.

To explain the above phenomenon and obtain the sharp estimates we shall decompose the multiplier dyadically with respect to the boundary of the cylinder and the cone $\{|\xi_{d+1}| = |\xi'|\}$. This was developed by P. Taylor to treat the case $p>1$ in [10]. This decomposition has both advantage and disadvantage. The disadvantage is that after analyzing each piece one has to add pieces together to recover the original multiplier so it is hard to obtain a precise asymptotic behavior of the kernel because of the possible cancellation which might occur during the summation. However it turns out that the decay estimates for the kernel function obtained in this way are sharp. The advantage is that it is easy to see which part of the multiplier makes a contribution to determining the critical index and the restriction on the range of p for which the estimates hold. One can easily see these during the proof of Lemma 1.

We shall prove the following theorems:

Theorem 1 *If* $4/5 < p < 1$ *and* $\lambda > 3(1/p-1)+1/2$, *then* T_*^λ *is bounded from* $H^p(\mathbb{R}^3)$ *to* $L^p(\mathbb{R}^3)$; *that is*

$$\left|\left|T_*^\lambda f\right|\right|_{L^p(\mathbb{R}^3)} \leq C\, ||f||_{H^p(\mathbb{R}^3)},$$

where the constant C *does not depend on* f.

Theorem 2 *If* $4/5 < p < 1$ *and* $\lambda = 3(1/p-1)+1/2$ *or* $p = 4/5$ *and* $\lambda > 3(1/p-1)+1/2$, *then* T_*^λ *maps* $H^p(\mathbb{R}^3)$ *boundedly into weak-*$L^p(\mathbb{R}^3)$, *that is,*

$$\left|\left\{(x,t)\in\mathbb{R}^2\times\mathbb{R} : T_*^\lambda f(x,t) > \alpha\right\}\right| \leq C\,\alpha^{-p}\, ||f||^p_{H^p(\mathbb{R}^3)}$$

where the constant C does not depend on α or f, and $|B|$ denotes the Lebesgue measure of B.

In Section 2 we describe the decomposition of the multiplier to obtain the decay of the kernel function. In Section 3 we shall prove the theorems based on the decay of the kernel. In Section 4 we prove that the results stated above are sharp.

Remark 1 (i) Let $\lambda_p = 3(1/p-1)+1/2$ be the critical index. If $\lambda < \lambda_p$ or $p \le 4/5$, it will be shown that even T^λ fails to be bounded on $L^p(\mathbb{R}^3)$ (see Section 4). The indicated ranges of p and λ can not be improved.
(ii) We do not know whether T_1^λ maps $H^{4/5}(\mathbb{R}^3)$ boundedly into weak-$L^{4/5}(\mathbb{R}^3)$ when $\lambda = \lambda_p$.
(iii) It is not known whether $T_*^{\frac{1}{2}}$ is of weak type $(1,1)$ or just of weak type $(1,1)$ on functions in $H^1(\mathbb{R}^3)$.
(iv) In the cases of dimensions $d \ge 3$ on $\mathbb{R}^d \times \mathbb{R}$, a critical index in the above sense does not exist because $(1-\rho)_+^\lambda$ is not integrable independently of λ (see [2,10] for details).

In what follows we write $A \lesssim B$ if there is a positive constant C such that $A \le CB$.

2 Kernel estimates

In this section we shall obtain the decay estimates for the kernel of T_ϵ^λ by using a suitable dyadic decomposition of the multiplier. We first write

$$T_\epsilon^\lambda f(x,t) = (2\pi)^{-3} \int_{\mathbb{R}} \int_{\mathbb{R}^2} G^\epsilon(x-y, t-s) f(y,s)\, dy\, ds$$

where

$$G^\epsilon(x,t) = \int_{\mathbb{R}} \int_{\mathbb{R}^2} \big(1 - \frac{\rho(\xi)}{\epsilon}\big)_+^\lambda e^{i<x,\xi'>+it\xi_3}\, d\xi'\, d\xi_3.$$

We note that G^ϵ satisfies the following dilation property

$$G^\epsilon(\cdot,\cdot) = \epsilon^3 G^1(\epsilon\cdot, \epsilon\cdot). \tag{2.1}$$

To simplify notations we set $G = G^1$. Let $\phi \in C_0^\infty(\mathbb{R})$ be supported in $[1/2, 2]$ such that $\sum_{k=2}^\infty \phi(2^k s) = 1$ for $s \in (0, 1/4)$ and set $\phi_1(s) = \chi_{[0,1)}(s) - \sum_{k=2}^\infty \phi(2^k(1-s))$.

We decompose the multiplier $(1-\rho(\cdot))_+^\lambda$ into several types of dyadic pieces defined by

$$m_{jk}(\xi) = \phi(2^j(1-|\xi'|))\phi(2^k(1-|\xi_3|))(1-\rho(\xi))_+^\lambda \quad \text{if } j \ge 2 \text{ and } k \ge 2, \tag{2.2}$$
$$m_{1k}(\xi) = \phi_1(|\xi'|)\phi(2^k(1-|\xi_3|))(1-\rho(\xi))_+^\lambda \quad \text{if } k \ge 2,$$
$$m_{j1}(\xi) = \phi_1(|\xi_3|)\phi(2^j(1-|\xi'|))(1-\rho(\xi))_+^\lambda \quad \text{if } j \ge 2,$$
$$m_{11}(\xi) = (1-\rho(\xi))^\lambda \phi_1(|\xi'|)\phi_1(|\xi_3|).$$

We write

$$G_{j,k} = \mathcal{F}^{-1}[m_{jk}] \quad \text{and} \quad G = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} G_{j,k}$$

where $\mathcal{F}^{-1}[f]$ is denoted by the inverse Fourier transform of f. Since the multiplier has symmetry with respect to ξ' the kernel can be expressed by the Bessel function J_μ of order $\mu > -\frac{1}{2}$ defined by

$$\text{(i)} \quad J_\mu(t) = A_\mu t^\mu \int_{-1}^{1} e^{it\sigma}(1-\sigma^2)^{\mu-\frac{1}{2}}\, d\sigma \quad \text{where } A_\mu = \left[2^\mu \Gamma(2\mu+1)\Gamma(1/2)\right]^{-1}, \tag{2.3}$$

$$\text{(ii)} \quad \frac{d}{dt}\{t^{-\mu} J_\mu(t)\} = -t\{t^{-(\mu+1)} J_{\mu+1}(t)\}.$$

We refer to [7] for more detailed properties of J_μ.

For further decomposition of the kernel we shall use the idea of Müller and Seeger in [3]. They used dyadic decomposition of Bessel function to prove local smoothing conjecture for spherically symmetric initial data including endpoint results. Let $\eta \in C_0^\infty(\mathbb{R})$ be supported in $(-1/2, 2)$ and equal to 1 in $(-1/4, 1/4)$. For $m = 0, 1, 2, \ldots$ we set

$$\eta_m(\sigma,\nu) = \begin{cases} \eta(\nu(1-\sigma^2)) & \text{if } m = 0 \\ \eta(2^{-m}\nu(1-\sigma^2)) - \eta(2^{-m+1}\nu(1-\sigma^2)) & \text{if } m > 0 \end{cases}$$

and

$$J_\mu^m(u\nu) = A_\mu (u\nu)^\mu \int_{-1}^{1} e^{i(u\nu)\sigma}\,(1-\sigma^2)^{\mu-1/2}\eta_m(\sigma,\nu)\, d\sigma.$$

For a positive integer M we define

$$\phi_{m\nu}(\sigma) = \begin{cases} (1-\sigma^2)^{\mu-1/2}\,\eta_m(\sigma,\nu) & \text{if } m = 0 \\ \left(\frac{1}{iu\nu}\right)^M \left(\frac{d}{d\sigma}\right)^M [\eta_m(\sigma,\nu)(1-\sigma^2)^{\mu-1/2}] & \text{if } m > 0. \end{cases}$$

Then by integration by parts if $m > 0$ we have

$$J_\mu^m(u\nu) = A_\mu (u\nu)^\mu \int_{-1}^{1} e^{i(u\nu)\sigma}\phi_{m\nu}(\sigma)\, d\sigma. \tag{2.4}$$

We note that the integrand in (2.4) has the following upper bound:

$$|\phi_{m\nu}(\sigma)| \le C\, u^{-M} 2^{-mM}\, (2^m \nu^{-1})^{\mu-1/2} \tag{2.5}$$

and that $\phi_{m\nu}$ vanishes unless either $1-\sigma^2 \approx 2^m\nu^{-1}$ for $m > 0$, or $1-\sigma^2 \le \nu^{-1}$ for $m = 0$ so if σ is in the support of $\phi_{m\nu}$ then either $|\nu - \nu\sigma| \le 2^m$ or $|\nu+\nu\sigma| \le 2^m$.

Now we return to discussion on the kernel. Since the kernel and each piece of its decomposition are symmetric in x variable, there exist functions $K_{j,k}$ in $\mathbb{R}^2$ with which we can write $G_{j,k}(x,t) = K_{j,k}(|x|, t)$. We define

$$K = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} K_{j,k}.$$

Lemma 1 *For a positive integer N*

$$|K(r,t)| + \sum_{|\gamma|=N} |K^{(\gamma)}(r,t)| \le \frac{C}{(1+r)^{\lambda+\frac{3}{2}}(1+|t|)} \chi_{\{r\ge|t|\}}(r,t) + \frac{C}{(1+r)^{\frac{3}{2}}(1+|t|)^{\lambda+1}} \chi_{\{r\le|t|\}}(r,t) + \frac{C}{(1+r)^2(1+|t|)^2} \chi_{\{r\le|t|\}}(r,t) + \frac{C}{(1+r)^{5/2}(1+||t|-r|)^{\lambda}} \chi_{\{2r\ge|t|\ge r/2\}}(r,t) \tag{2.6}$$

where $\gamma = (\gamma_1, \gamma_2)$ *is a pair of non-negative integers with* $|\gamma| := \gamma_1 + \gamma_2$ *and* χ_A *is the characteristic function of A.*

Proof We fix $r > 0$ and t. We first consider the case $j \ge k+2$ and $k \ge 2$. In this case $\rho(\xi) = |\xi'|$ in (2.2). We set $s = |\xi'|$ and use the Bessel function J_0 to write

$$\begin{aligned} &G_{j,k}(x,t) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^2} \phi(2^j(1-|\xi'|))\,(1-|\xi'|)^{\lambda}\phi(2^k(1-\xi_3))\, e^{i<x,\xi'>+it\xi_3}\, d\xi'\, d\xi_3 \\ &= \int_{\mathbb{R}^2} \phi(2^j(1-|\xi'|))\,(1-|\xi'|)^{\lambda} e^{i<x,\xi'>}\, d\xi' \int_{\mathbb{R}} \phi(2^k(1-\xi_3)) e^{it\xi_3}\, d\xi_3. \end{aligned}$$

By Bochner's formula in [8] we have

$$K_{j,k}(r,t) = \int_0^{\infty} J_0(rs)\phi(2^j(1-s))\,(1-s)^{\lambda}\, s\, ds \int_{\mathbb{R}} \phi(2^k(1-\xi_3)) e^{it\xi_3}\, d\xi_3.$$

By using (2.4), we decompose

$$K_{j,k} = \sum_{m=0}^{\infty} K_{j,k}^m$$

where

$$\begin{aligned} &K_{j,k}^m(r,t) \\ &= \int_0^{\infty} J_0^m(rs)\phi(2^j(1-s))\,(1-s)^{\lambda}\, s\, ds \int_{\mathbb{R}} \phi(2^k(1-\xi_3)) e^{it\xi_3}\, d\xi_3 \\ &= A_0 \int_{-1}^{1} \phi_{mr}(\sigma) \int_0^{\infty} e^{irs\sigma} \phi(2^j(1-s))\,(1-s)^{\lambda}\, s\, ds \int_{\mathbb{R}} \phi(2^k(1-\xi_3)) e^{it\xi_3}\, d\xi_3. \end{aligned} \tag{2.7}$$

We integrate by parts with respect to s and ξ_3 in (2.7) and apply the Fubini theorem to obtain

$$\begin{aligned} &|K_{j,k}^m(r,t)| \\ &\le C \int_{-1}^{1}\int_0^{1} |\phi_{mr}(\sigma)|\,(1+|\sigma r|)^{-N} \left| \left(\frac{\partial}{\partial s}\right)^{N} \varphi(2^j(1-s))(1-s)^{\lambda} s \right| ds\, d\sigma \\ &\quad \times \int_{\mathbb{R}} (1+|t|)^{-N_1} \left| \left(\frac{\partial}{\partial \xi_3}\right)^{N_1} \phi(2^k(1-|\xi_3|)) \right| d\xi_3. \end{aligned} \tag{2.8}$$

Then in view of (2.5), the integrand of (2.8) is bounded by

$$C\,2^{-j\lambda}\,|\phi_{mr}(\sigma)|\frac{1}{(1+2^{-j}|\sigma r|)^N}\,\frac{1}{(1+2^{-k}|t|)^{N_1}}$$
$$\le C\,2^{-j\lambda}\,2^{m(-1/2+N-M)}\,\frac{r^{1/2}}{(1+2^{-j}r)^N}\,\frac{1}{(1+2^{-k}|t|)^{N_1}}.$$

Due to the domain of the integration we gain an additional factor of $C\,2^m r^{-1}\,2^{-k}\,2^{-j}$. Since we can take M so that $M > N+1/2$, we sum over m for $K^m_{j,k}$ to obtain

$$|K_{j,k}(r,t)| \;\le\; C\,2^{-k}\,2^{-j(\lambda+1)}\,\frac{r^{-1/2}}{(1+2^{-j}r)^N}\,\frac{1}{(1+2^{-k}|t|)^{N_1}}.$$

If $r\ge|t|$ then we sum over j and k to obtain

$$\sum_{j\ge k+2,k\ge 2}|K_{j,k}(r,t)| \lesssim \Big(\sum_{2^{-j}r\le 1}2^{-j(\lambda+1)}\,r^{-1/2} + \sum_{2^{-j}r>1}2^{-j(\lambda+1-N)}\,r^{-(N+1/2)}\Big)$$
$$\times\Big(\sum_{2^{-k}|t|\le 1}2^{-k} + \sum_{2^{-k}|t|>1}2^{-k(1-N_1)}\,|t|^{-N_1}\Big)$$
$$\lesssim \frac{1}{(1+r)^{\lambda+\frac{3}{2}}(1+|t|)}.$$

If $r\le|t|$ then we use $j\ge k+2$ to obtain

$$\sum_{j\ge k+2,\,k\ge 2}|K_{j,k}(r,t)| \lesssim \Big(\sum_{2^{-j}r\le 1}2^{-j}\,r^{-1/2} + \sum_{2^{-j}r>1}2^{-j(1-N)}\,r^{-(N+1/2)}\Big)$$
$$\times\Big(\sum_{2^{-k}|t|\le 1}2^{-k(1+\lambda)} + \sum_{2^{-k}|t|>1}2^{-k(1+\lambda-N_1)}\,|t|^{-N_1}\Big)$$
$$\lesssim \frac{1}{(1+r)^{\frac{3}{2}}(1+|t|)^{1+\lambda}}.$$

Thus we have obtained

$$\sum_{j\ge k+2,\,k\ge 2}|K_{j,k}(r,t)|$$
$$\le \frac{C}{(1+r)^{\lambda+\frac{3}{2}}(1+|t|)}\chi_{\{r\ge|t|\}}(r,t) + \frac{C}{(1+r)^{\frac{3}{2}}(1+|t|)^{\lambda+1}}\chi_{\{r\le|t|\}}(r,t). \tag{2.9}$$

Now we consider the case $k\ge j+2$ and $j\ge 2$. Since $\rho(\xi)=|\xi_3|$ in this case, we can write

$$K^m_{j,k}(r,t) = A_0\int_{-1}^{1}\int_0^1 \phi_{mr}(\sigma)e^{irs\sigma}\phi(2^j(1-s))\,s ds\,d\sigma$$
$$\times\int_{\mathbb{R}} e^{it\xi_3}\phi(2^k(1-|\xi_3|))(1-|\xi_3|)^\lambda\,d\xi_3. \tag{2.10}$$

We integrate by parts with respect to s and ξ_3 and apply Fubini's theorem and (2.5) to obtain that the integrand in (2.10) is bounded by

$$\begin{aligned}
&C\,2^{-k\lambda}\,|\phi_{mr}(\sigma)|\frac{1}{(1+2^{-j}|\sigma r|)^N}\,\frac{1}{(1+2^{-k}|t|)^{N_1}}\\
&\quad\le C\,2^{m(-1/2+N-M)}\,\frac{r^{1/2}}{(1+2^{-j}r)^N}\,\frac{2^{-k\lambda}}{(1+2^{-k}|t|)^{N_1}}.
\end{aligned}$$

In view of the domain of the integration we gain the factor $C2^m r^{-1}2^{-k}2^{-j}$ after the integration. We choose $M > N + 1/2$ and sum over m to obtain

$$|K_{j,k}(r,t)| \;\le\; C\,2^{-j}\,\frac{r^{-1/2}}{(1+2^{-j}r)^N}\,\frac{2^{-k(\lambda+1)}}{(1+2^{-k}|t|)^{N_1}}.$$

Now we use the same argument as above and combine with (2.9) to obtain

$$\begin{aligned}
&\sum_{|j-k|\ge 2,\, j,\,k\ge 2} |K_{j,k}(r,t)|\\
&\le \frac{C}{(1+r)^{\lambda+\frac{3}{2}}(1+|t|)}\chi_{\{r\ge|t|\}}(r,t) + \frac{C}{(1+r)^{\frac{3}{2}}(1+|t|)^{\lambda+1}}\chi_{\{r\le|t|\}}(r,t).
\end{aligned} \tag{2.11}$$

The cases $j\ge 2,\ k=1$ and $k\ge 2,\ j=1$ can be treated similarly as above, we omit the detail and leave it to the interested readers.

Now we take into account of the case $|k-j|\le 1$ and $k,\ j\ge 2$. In this case we define a new multiplier μ_{jk} by subtracting a harmless term:

$$\mu_{jk}(\xi) = m_{jk}(\xi) - (1-|\xi'|)^{\lambda}\phi(2^j(1-|\xi'|))\phi(2^k(1-|\xi_3|)).$$

By the same argument as above it is easy to see that $\sum_{|j-k|\le 1,\ j,k\ge 2} |\mathcal{F}^{-1}[m_{jk} - \mu_{jk}]|$ satisfies decay estimates in (2.11). By the definition $\mu_{jk}(\xi) = 0$ if $|\xi_3|\le|\xi'|$. If $|\xi_3|\ge|\xi'|$, then

$$\begin{aligned}
\mu_{jk}(\xi) &= [(1-|\xi_3|)^{\lambda} - (1-|\xi'|)^{\lambda}]\phi(2^j(1-|\xi'|))\phi(2^k(1-|\xi_3|))\\
&= |\xi_3|\left(1-\frac{|\xi'|}{|\xi_3|}\right)\phi(2^j(1-|\xi'|))\phi(2^k(1-|\xi_3|))\\
&\qquad\times\int_0^1 \lambda(1-t|\xi_3|-(1-t)|\xi'|)^{\lambda-1}\,dt,
\end{aligned}$$

and so it behaves like the cone multiplier in $\mathbb{R}^2\times\mathbb{R}$. Based on this observation we make a further decomposition by using the distance from the cone. Without loss of generality we may assume that $\xi_3\ge 0$. We fix $k,\ j$, and then define μ_{jkl} and $L_{j,k,l}$ by

$$\mu_{jkl}(\xi) = \phi\left(2^l\left(1-\frac{|\xi'|}{\xi_3}\right)\right)\mu_{jk}(\xi) \quad\text{and}\quad L_{j,k,l} = \mathcal{F}^{-1}\big[\mu_{jkl}\big].$$

Since if $l < j - 6$

$$\text{supp}\,\phi\left(2^l\left(1 - \frac{|\xi'|}{\xi_3}\right)\right) \cap \text{ supp}\,\mu_{jk} = \emptyset,$$

we may assume that $l \geq j - 6$. We decompose

$$L_{j,k,l} = \sum_{m=0}^{\infty} L^m_{j,k,l}$$

where

$$\begin{aligned} L^m_{j,k,l}(r,t) &= \int_{\mathbb{R}}\int_0^1 J_0^m(rs\xi_3)q_{jkl}(s,\xi_3)\, s\,\xi_3^2\, e^{it\xi_3}\, ds\, d\xi_3 \\ &= A_0 \int_{-1}^1 \phi_{mr}(\sigma)\int_{\mathbb{R}}\int_0^1 q_{jkl}(s,\xi_3)\, s\,\xi_3^2\, e^{irs\sigma\xi_3 + it\xi_3}\, ds\, d\xi_3\, d\sigma, \end{aligned}$$

with

$$q_{jkl}(s,\xi_3) = \phi(2^l(1-s))\,\phi(2^j(1-s\xi_3))\,\phi(2^k(1-\xi_3))[(1-\xi_3)^\lambda - (1-s\xi_3)^\lambda].$$

By integration by parts with respect to s and ξ_3 and Fubini's theorem, we obtain

$$\begin{aligned} |L^m_{j,k,l}(r,t)| \leq C &\int_{\mathbb{R}}\int_{-1}^1\int_0^1 |\phi_{mr}(\sigma)|\,(1+|\sigma r|)^{-N}\,(1+|t+rs\sigma|)^{-N_1} \\ &\times \left|\,\left(\frac{\partial}{\partial\xi_3}\right)^{N_1}\left(\frac{\partial}{\partial s}\right)^N q_{jkl}(s,\xi_3)\, s\,\xi_3^2\right| ds\, d\sigma\, d\xi_3. \end{aligned} \tag{2.12}$$

Due to the estimate (2.5) with the assumption $|k-j| \leq 1$ and $k,\ j \geq 2$, the integrand of (2.12) is bounded by

$$\begin{aligned} &C\,2^{-j(\lambda-1)}2^{-l}|\phi_{mr}(\sigma)|\frac{1}{(1+2^{-l}|\sigma r|)^N}\,\frac{1}{(1+2^{-k}|t+rs\sigma|)^{N_1}} \\ &\leq C\,2^{-j(\lambda-1)}2^{-l}\,2^{m(-1/2+N+N_1-M)}\,r^{1/2} \\ &\quad\times \frac{1}{(1+2^{-l}r)^N}\left\{\frac{1}{(1+2^{-k}|t+rs|)^{N_1}} + \frac{1}{(1+2^{-k}|t-rs|)^{N_1}}\right\}. \end{aligned}$$

If we integrate $\{\frac{1}{(1+2^{-k}|t+rs|)^{N_1}} + \frac{1}{(1+2^{-k}|t-rs|)^{N_1}}\}$ over the support of $\phi(2^l(1-s)) \otimes \phi_{mr} \otimes \phi(2^k(1-|\xi_3|))$ for $m \geq 0$, we get an additional factor of $C\,2^m r^{-1}\,2^{-k}\,2^{-l}$ with

$$\left\{\frac{1}{(1+2^{-k}|t+r|)^{N_1}} + \frac{1}{(1+2^{-k}|t-r|)^{N_1}}\right\}.$$

By $M > N + N_1 + 1/2$, we can sum over m and l to obtain

$$\begin{aligned} |L_{j,k,l}(r,t)| &\leq \sum_{m=0}^{\infty} |L^m_{j,k,l}(r,t)| \\ &\leq C\,r^{-1/2}\,2^{-2l}2^{-j\lambda}\min\{1,(2^{-l}r)^{-N}\}\min\{1,(2^{-k}||t|-r|)^{-N_1}\}. \end{aligned}$$

If $|t| \geq 2r$, then

$$\sum_l \sum_{|j-k|\leq 1, j,k\geq 2} |L_{j,k,l}(r,t)| \leq \sum_l \sum_{|j-k|\leq 1, j,k\geq 2} Cr^{-1/2} 2^{-l} 2^{-k} 2^{-k\lambda} \min\{1, (2^{-l} r)^{-N}\} \times \min\{1, (2^{-k}||t|-r|)^{-N_1}\} \leq \frac{C}{r^{3/2}||t|-r|^{\lambda+1}},$$

which implies

$$\sum_l \sum_{|j-k|\leq 1,\, j,\, k\geq 2} |L_{j,k,l}(r,t)| \leq \frac{C}{(1+r)^{3/2}(1+|t|)^{\lambda+1}} \chi_{\{|t|\geq 2r\}}.$$

If $|t| \leq 2r$, then

$$\sum_l \sum_{|j-k|\leq 1,\, j,\, k\geq 2} |L_{j,k}(r,t)| \leq \frac{C}{(1+r)^{5/2}(1+||t|-r|)^{\lambda}} \chi_{\{|t|\leq 2r\}}. \tag{2.13}$$

Lastly, we consider the case $k,\ j = 1$. In this case we also subtract by multipliers which have nice decay. The multipliers we subtract by are determined by the size of r and $|t|$.

If $2r \leq |t|$, then we define

$$\mu_{11}(\xi) = m_{11}(\xi) - (1 - |\xi'|)^{\lambda} \phi_1(\xi_3)\phi_1(|\xi'|).$$

It is easy to see that $\mathcal{F}^{-1}[m_{11} - \mu_{11}]$ has fast decay and so it suffices to consider μ_{11}.

We define

$$\mu_{11l}(\xi) = \begin{cases} \phi\big(2^l\big(1 - \frac{|\xi'|}{\xi_3}\big)\big)\ \mu_{11}(\xi) & \text{if } \ l \geq 2 \\ \phi_1\big(\frac{|\xi'|}{\xi_3}\big)\ \mu_{11}(\xi) & \text{if } \ l = 1 \end{cases}$$

and $L_{1,1,l} = F^{-1}\big[\mu_{11l}\big]$ for $l = 1, 2, ...$ We first assume $l \geq 2$. We then have $|\xi'| \approx \xi_3$. For the complete control of sizes of ξ_3 and $|\xi'|$ we decompose

$$L_{1,1,l} = \sum_{k=0}^{\infty} L_{1,1,l,k}$$

where

$$L_{1,1,l,k}(r,t) = \int_{\mathbb{R}} \int_{\mathbb{R}^2} J_0(rs\xi_3)\phi(2^l(1-s))\,[(1-\xi_3)^{\lambda} - (1-s\xi_3)^{\lambda}]\,s \times \phi_1(s\xi_3)\,\phi_1(\xi_3)\,\phi(2^k\xi_3)\,\xi_3^2\,e^{it\xi_3}\,ds\,d\xi_3.$$

Similarly as before, we decompose

$$L_{1,1,l,k} = \sum_{m=0}^{\infty} L^m_{1,1,l,k}$$

where

$$L^m_{1,1,l,k}(r,t) = \int_{\mathbb{R}}\int_{\mathbb{R}^2} J_0^m(rs\xi_3)\phi(2^l(1-s))\,[(1-\xi_3)^\lambda - (1-s\xi_3)^\lambda]\,s \\ \times \phi_1(s\xi_3)\,\phi_1(\xi_3)\,\phi(2^k\xi_3)\,\xi_3^2\,e^{it\xi_3}\,ds\,d\xi_3.$$

We use (2.4) to integrate by parts with respect to s and ξ_3 and then apply Fubini's theorem to obtain

$$\begin{aligned} &|L^m_{1,1,l,k}(r,t)| \\ &\le C \int_{\mathbb{R}}\int_{-1}^{1}\int_0^1 |\phi_{m(2^{-k}r)}(\sigma)|\,(1+|\sigma\xi_3 r|)^{-N}\,(1+|t+rs\sigma|)^{-N_1} \\ &\quad \times \Big|\Big(\frac{\partial}{\partial\xi_3}\Big)^{N_1}\Big(\frac{\partial}{\partial s}\Big)^{N}\phi(2^l(1-s))\,[(1-\xi_3)^\lambda-(1-s\xi_3)^\lambda]\,s \\ &\quad \times \phi_1(s\xi_3)\,\phi_1(\xi_3)\,\phi(2^k\xi_3)\,\xi_3^2\Big|\,ds\,d\sigma\,d\xi_3. \end{aligned} \tag{2.14}$$

We observe

$$|(1-\xi_3)^\lambda - (1-s\xi_3)^\lambda| \le C\xi_3|1-s| \le C2^{-k}2^{-l}$$

and use (2.5) and the domain of the integration in (2.14) to obtain

$$\begin{aligned} &|L_{1,1,l,k}(r,t)| \\ &\le C\,r^{-1/2}\sum_{m=0}^{\infty} 2^{m(N+N_1-M+1/2)}2^{-2l}2^{-7k/2}\frac{1}{(1+2^{-l-k}r)^N}\,\frac{1}{(1+2^{-k}||t|-r|)^{N_1}} \\ &\le Cr^{-1/2}2^{-2l}2^{-7k/2}\frac{1}{(1+2^{-l-k}r)^N}\,\frac{1}{(1+2^{-k}||t|-r|)^{N_1}} \end{aligned}$$

where $N > N_1 + 4$ and $M > N + N_1 + 1/2$.

Since $r < ||t|-r|$, we obtain

$$\begin{aligned} \Big|\sum_{l\ge2} L_{1,1,l}(r,t)\Big| &\le \sum_{k\ge1}\sum_{l\ge2}|L_{1,1,l,k}(r,t)| \\ &\le C\sum_{k\ge1}\sum_{l\ge2} r^{-1/2}2^{-2l}2^{-7k/2}\frac{1}{(1+2^{-k}||t|-r|)^{N_1}} \\ &\le Cr^{-1/2}\sum_{k\ge1}2^{-7k/2}\frac{1}{(1+2^{-k}||t|-r|)^{N_1}} \\ &\le Cr^{-1/2}||t|-r|^{-7/2}. \end{aligned} \tag{2.15}$$

To treat $L_{1,1,1}$ we decompose

$$L_{1,1,1} = \sum_{j \geq k,\, k \geq 1} \sum_{m=0} L^m_{1,1,1,j,k}$$

where

$$L^m_{1,1,1,j,k}(r,t) = \int_{\mathbb{R}}\int_{\mathbb{R}^2} J_0^m(rs)\phi_1(s/\xi_3)\,[(1-\xi_3)^\lambda - (1-s)^\lambda]\,s \times \phi_1(s)\,\phi_1(\xi_3)\,\phi(2^k\xi_3)\,\phi(2^j s)\,e^{it\xi_3}\,ds\,d\xi_3.$$

By the integration by parts and using similar arguments as above we obtain

$$|L^m_{1,1,1,j,k}(r,t)| \leq C2^{m(N+N_1-M+1/2)} r^{-1/2} 2^{j/2} 2^{-2j} 2^{-2k} \frac{1}{(1+2^{-j}r)^N (1+2^{-k}|t|)^{N_1}}$$

where $M > N + N_1 + 1/2$. By summing in m, j, and k we obtain

$$|L_{1,1,1}(r,t)| \leq \frac{C}{(1+r)^2(1+|t|)^2}\chi_{\{|t| \geq 2r\}}(r,t).$$

If $r \geq ||t| - r|$, then we define

$$\tilde{\mu}_{11}(\xi) = m_{11}(\xi) - (1-|\xi_3|)^\lambda \phi_1(\xi_3)\phi_1(|\xi'|).$$

It is easy to see that $\mathcal{F}^{-1}[m_{11} - \tilde{\mu}_{11}]$ has fast decay so it suffices to consider $\tilde{\mu}_{11}$. We define

$$\tilde{\mu}_{11l}(\xi) = \begin{cases} \phi\big(2^l\big(\frac{|\xi'|}{\xi_3} - 1\big)\big)\ \mu_{11}(\xi) & \text{if} \quad l \geq 2 \\ \phi_1\big(\frac{\xi_3}{|\xi'|}\big)\ \mu_{11}(\xi) & \text{if} \quad l = 1 \end{cases}$$

and $\tilde{L}_{1,1,l} = \mathcal{F}^{-1}\big[\tilde{\mu}_{11l}\big]$ for $l = 1, 2, \ldots$. As above we use $|\xi_3| \approx 2^{-k}$ to make a further decomposition

$$\tilde{L}_{1,1,l} = \sum_{k=1}^{\infty} \tilde{L}_{1,1,l,k}.$$

When $l \geq 2$ by using the same argument as above it is easy to check that $\tilde{L}_{1,1,l,k}$ has the same pointwise estimates as $L_{1,1,l,k}$. Since $2r > |t|$, we have

$$\begin{aligned} &\left|\sum_{l \geq 2} \tilde{L}_{1,1,l}(r,t)\right| \qquad (2.16) \\ &\leq \sum_{k \geq 1}\sum_{l \geq 2} |\tilde{L}_{1,1,l,k}(r,t)| \\ &\leq \sum_{k \geq 1}\left(\sum_{2^l \geq 2^{-k}r} + \sum_{2^l < 2^{-k}r} |\tilde{L}_{1,1,l,k}(r,t)|\right) \\ &\leq Cr^{-1/2}\sum_{k \geq 1} 2^{2k} r^{-2} 2^{-7k/2} \frac{1}{(1+2^{-k}||t|-r|)^{N_1}} \\ &\leq Cr^{-5/2}||t|-r|^{-3/2}. \end{aligned}$$

By combining (2.13) with (2.15) and (2.16) we obtain

$$\begin{aligned}
&\sum_{|k-j|\le 1,\, j,\, k\ge 2}^{\infty} |L_{j,k}(r,t)| + \sum_{l\ge 2}(|L_{1,1,l}(r,t)| + |\tilde{L}_{1,1,l}(r,t)|) \\
&\le \frac{C}{(1+r)^{5/2}(1+||t|-r|)^{\lambda}}\chi_{\{2r\ge|t|\ge r/2\}}(r,t) \\
&\quad + \frac{C}{(1+r)^{3/2}(1+|t|)^{\lambda+1}}\chi_{\{2r\le|t|\}}(r,t) \\
&\quad + \frac{C}{(1+r)^{3/2+\lambda}(1+|t|)}\chi_{\{r\ge 2|t|\}}(r,t).
\end{aligned}$$

The treatment of $\tilde{L}_{1,1,1}$ is similar as that of $L_{1,1,1}$. Actually for $\tilde{L}_{1,1,1}$ we obtain

$$|\tilde{L}_{1,1,1}(r,t)| \le \frac{C}{(1+r)^3(1+|t|)}\chi_{\{2r\ge|t|\}}(r,t).$$

We have proved (2.6) with $\gamma = (0,0)$. The estimates for the derivatives of the kernel can be obtained by the same argument except that we apply (2.3)-(ii). We omit the details. We therefore complete the proof of Lemma 1. □

3 Proofs of theorems

In this section we prove Theorem 1 and Theorem 2. To do this we define Hardy spaces.

Definition 1 Let $0 < p \le 1$ and s be an integer that satisfies $s \ge 3(1/p - 1)$. Let Q be a cube in $\mathbb{R}^3$. We say that $\mathfrak{a}$ is a (p,s)-atom associated with Q if $\mathfrak{a}$ is supported on $Q \subset \mathbb{R}^3$ and satisfies

$$\begin{aligned}
&(i) \quad ||\mathfrak{a}||_{L^\infty(\mathbb{R}^3)} \le |Q|^{-1/p}; \\
&(ii) \quad \int_{\mathbb{R}^3} \mathfrak{a}(x)\, x^{\beta}\, dx = 0,
\end{aligned}$$

where $\beta = (\beta_1, \beta_2, \beta_3)$ is an 3-tuple of non-negative integers satisfying $|\beta| \le \beta_1 + \beta_2 + \beta_3 \le s$, and $x^{\beta} = x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3}$.

If $\{\mathfrak{a}_j\}$ is a collection of (p,s)-atoms and $\{c_j\}$ is a sequence of complex numbers with $\sum_{j=1}^{\infty} |c_j|^p < \infty$, then the series $f = \sum_{j=1}^{\infty} c_j \mathfrak{a}_j$ converges in the sense of distributions, and its sum belongs to H^p with the quasinorm (see [7])

$$||f||_{H^p} = \inf_{\sum_{j=1}^{\infty} c_j \mathfrak{a}_j = f} \Big(\sum_{j=1}^{\infty} |c_j|^p\Big)^{1/p}.$$

To prove Theorem 2 we shall need a lemma by Stein, Taibleson and Weiss.

Lemma 2 *Suppose* $0 < p < 1$ *and* $\{f_j\}$ *is a sequence of measurable functions such that*

$$|\{x : |f_j(x)| > \alpha > 0\}| \leq \alpha^{-p} \tag{3.1}$$

for $j = 1, 2, 3, \ldots$. *If* $\sum_{j=1}^{\infty} |c_j|^p \leq 1$, *then*

$$\left|\left\{x : \left|\sum_{j=1}^{\infty} c_j f_j(x)\right| > \alpha\right\}\right| \leq \frac{2-p}{1-p}\, \alpha^{-p}.$$

Proof See Lemma 1.8 in [9]. □

We shall use the following elementary lemma to obtain weak type estimates in the proof of Proposition 1.

Lemma 3 *Let* $\mathfrak{a}$, $\mathfrak{b}$, c_1, *and* $p < 1$ *be positive real numbers.*
If $\mathfrak{a} + \mathfrak{b} = \frac{3}{p}$ *and* $\mathfrak{b} < \frac{1}{p}$, *that is,* $\mathfrak{a} > \frac{2}{p}$, *then*

$$\begin{aligned} &\left|\left\{(x,t) \,:\, c_1|x| \geq |t|,\ |x|^{-\mathfrak{a}}|t|^{-\mathfrak{b}} > \alpha/C\right\}\right| \\ &\quad + \left|\left\{(x,t) \,:\, |x| > 2,\ |t| \leq 4,\ |x|^{-2/p} > \alpha/C\right\}\right| \ \lesssim\ \alpha^{-p}. \end{aligned} \tag{3.2}$$

If $\mathfrak{a} + \mathfrak{b} = \frac{3}{p}$ *and* $\mathfrak{b} > \frac{1}{p}$, *that is,* $\mathfrak{a} < \frac{2}{p}$, *then*

$$\begin{aligned} &\left|\left\{(x,t) \,:\, c_1|x| \leq |t|,\ |x|^{-\mathfrak{a}}|t|^{-\mathfrak{b}} > \alpha/C\right\}\right| \\ &\quad + \left|\left\{(x,t) \,:\, |x| \leq 4,\ |t| > 2,\ |t|^{-1/p} > \alpha/C\right\}\right| \ \lesssim\ \alpha^{-p}. \end{aligned} \tag{3.3}$$

If $\mathfrak{a} + \mathfrak{b} > \frac{3}{p}$ *and* $\mathfrak{a} = \frac{2}{p}$, *then*

$$\left|\left\{(x,t) \,:\, |x| \geq c_1||t|-|x|| \geq 1,\ |x|^{-\mathfrak{a}}||t|-|x||^{-\mathfrak{b}} > \alpha/C\right\}\right| \lesssim \alpha^{-p}. \tag{3.4}$$

In addition, we have

$$\begin{aligned} &\left|\left\{(x,t) \,:\, \chi_{\{|x|\leq 4,\, |t|\leq 4\}}(x,t) > \alpha/C\right\}\right| \\ &\quad + \left|\left\{(x,t) \,:\, \chi_{\{|x|\leq 4,\, ||t|-|x||\leq 4\}}(x,t) > \alpha/C\right\}\right| \lesssim \alpha^{-p}. \end{aligned} \tag{3.5}$$

Proof The main idea of the proof is to find areas of the region defined by the inequality $|x|^{\mathfrak{a}}|t|^{\mathfrak{b}} \leq \alpha^{-1}$. For the integration we shall use the polar coordinates with $r = |x|$. If $\mathfrak{a} + \mathfrak{b} = \frac{3}{p}$ and $\mathfrak{b} < \frac{1}{p}$, then $\mathfrak{a}/\mathfrak{b} > 2$ so we obtain

$$\begin{aligned} &\left|\left\{(x,t) \,:\, c_1|x| \geq |t|,\ |x|^{-\mathfrak{a}}|t|^{-\mathfrak{b}} > \alpha/C\right\}\right| \\ &\lesssim \int_0^{\alpha^{-\frac{1}{\mathfrak{a}+\mathfrak{b}}}} \int_0^{c_1 r} dt\, r\, dr + \int_{\alpha^{-\frac{1}{\mathfrak{a}+\mathfrak{b}}}}^{\infty} \int_0^{\alpha^{-\frac{1}{\mathfrak{b}}} r^{-\frac{\mathfrak{a}}{\mathfrak{b}}}} dt\, r\, dr \\ &\lesssim \alpha^{-\frac{3}{\mathfrak{a}+\mathfrak{b}}} = \alpha^{-p}. \end{aligned}$$

Moreover, we immediately obtain

$$\left|\left\{(x,t)\,:\,|x|>2,\ |t|\le 4,\ |x|^{-2/p}>\alpha/C\right\}\right|\lesssim \alpha^{-p}\int_{|t|\le 4} dt.$$

We have proved (3.2).

If $\mathfrak{a}+\mathfrak{b}=\frac{3}{p}$ and $\mathfrak{b}>\frac{1}{p}$, then $2\mathfrak{b}/\mathfrak{a}>1$ so we apply exactly same argument as above with exchanging roles of $\mathfrak{a}$ and $\mathfrak{b}$, and t and r to prove (3.3). We omit the detail.

Now to prove (3.4) we assume that $\mathfrak{a}+\mathfrak{b}>\frac{3}{p}$ and $\mathfrak{a}=\frac{2}{p}$. By separately considering two cases: $t>0$ and $t<0$ and by using a change of variables $t'=t\pm|x|$ we may prove

$$\left|\left\{(x,t')\,:\,|x|\ge c_1|t'|\ge 1,\ |x|^{-\mathfrak{a}}|t'|^{-\mathfrak{b}}>\alpha/C\right\}\right|\lesssim \alpha^{-p}.$$

Since $2\mathfrak{b}/\mathfrak{a}>1$,

$$\left|\left\{(x,t')\,:\,c_1|x|\ge |t'|\ge 1,\ |x|^{-\mathfrak{a}}|t'|^{-\mathfrak{b}}>\alpha/C\right\}\right|\lesssim \alpha^{-p}\int_1^\infty \frac{1}{|t|^{2\mathfrak{b}/\mathfrak{a}}}\,dt,$$

which proves (3.4). Finally one can easily see that (3.5) is an immediate consequence of Chebyshev's inequality. □

To prove Theorem 2, we shall need uniform weak type estimates for T_*^λ with a (p,N)-atom $(N\ge 3(1/p-1))$.

Proposition 1 *Suppose f is a (p,N)-atom $(N\ge 3(1/p-1))$ on $\mathbb{R}^3$. Suppose $4/5<p<1$ and $\lambda=3(1/p-1)+1/2$ or $p=4/5$ and $\lambda>3(1/p-1)+1/2$. Then there exists a constant $C=C(p)$ such that*

$$|\{(x,t)\in\mathbb{R}^2\times\mathbb{R} : T_*^\lambda f(x,t)>\alpha\}|\le C\,\alpha^{-p} \tag{3.6}$$

for all $\alpha>0$.

Proof Let f be supported in a cube Q_0 of diameter 1 centered at the origin. We first consider the case $(x,t)\in Q_0^*$ which is the cube centered at the origin with diameter 2. In view of (2.1) and Lemma 1, we can easily see that G^ϵ is integrable and its L^1 norm is independent of ϵ. We have

$$|T_\epsilon^\lambda f(x,t)|\le ||G^\epsilon||_1||f||_\infty\le ||G^\epsilon||_1|Q_0^*|^{-1/p},$$

and therefore

$$T_*^\lambda f(x,t)=\sup_{\epsilon>0}|T_\epsilon^\lambda f(x,t)|\le C\,|Q_0^*|^{-1/p}$$

for all $(x,t)\in Q_0^*$. This implies that for $\alpha>0$

$$\left|\left\{(x,t)\in Q_0^* : T_*^\lambda f(x,t)>\alpha/C\right\}\right|\ \le\ C\,\alpha^{-p}. \tag{3.7}$$

Hence it suffices to show that for $\alpha>0$

$$\left|\left\{(x,t)\in\mathbb{R}^3\setminus Q_0^* : T_*^\lambda f(x,t)>\alpha/C\right\}\right|\ \le\ C\,\alpha^{-p}.$$

We fix $\epsilon \geq 1$ and use the fact that f is supported in Q_0 to write

$$|T_\epsilon^\lambda f(x,t)| \leq \epsilon^3 \iint_{Q_0} |f(y,s)|\,|K(\epsilon|x-y|, \epsilon(t-s))|\,dy\,ds.$$

By using the kernel estimates in Lemma 1 and (2.1), we write

$$\begin{aligned}
&|T_\epsilon^\lambda f(x,t)| \\
&\leq \epsilon^3 \iint_{Q_0} |f(y,s)| \Bigg[\frac{1}{(1+\epsilon|x-y|)^{\lambda+\frac{3}{2}}} \frac{1}{(1+\epsilon|t-s|)} \chi_{\{|x-y|\geq|t-s|\}} \\
&\quad + \frac{1}{(1+\epsilon|x-y|)^{\frac{3}{2}}} \frac{1}{(1+\epsilon|t-s|)^{\lambda+1}} \chi_{\{|x-y|\leq|t-s|\}} \\
&\quad + \frac{C}{(1+\epsilon|x-y|)^2 (1+\epsilon|t-s|)^2} \chi_{\{|x-y|\leq|t-s|\}} \\
&\quad + \frac{1}{(1+\epsilon|x-y|)^{\frac{5}{2}}} \frac{1}{(1+\epsilon||t-s|-|x-y||)^{\lambda}} \chi_{\{|x-y|/2\leq|t-s|\leq 2|x-y|\}} \Bigg] dy\,ds \\
&= T_{\epsilon,1}^\lambda f(x,t) + T_{\epsilon,2}^\lambda f(x,t) + T_{\epsilon,3}^\lambda f(x,t) + T_{\epsilon,4}^\lambda f(x,t).
\end{aligned} \tag{3.8}$$

If $|x| > 2$ and $|t| > 2$, then

$$\begin{aligned}
&|T_{\epsilon,1}^\lambda f(x,t)| + |T_{\epsilon,2}^\lambda f(x,t)| \\
&\leq C\,\epsilon^{(3-5/2-\lambda)} \left(|x|^{-(\lambda+3/2)}\, |t|^{-1}\, \chi_{\{4|x|\geq|t|\}} + |x|^{-3/2}\, |t|^{-(\lambda+1)}\, \chi_{\{|x|\leq 4|t|\}} \right).
\end{aligned} \tag{3.9}$$

If $|x| > 2$ and $|t| \leq 2$, then since $\lambda+3/2 = 2/p+2\delta_1$ where $2\delta_1 = 1/p-1 > 0$, we obtain

$$\begin{aligned}
&|T_{\epsilon,1}^\lambda f(x,t)| + |T_{\epsilon,2}^\lambda f(x,t)| \\
&\leq C\,\epsilon^{(3-2/p-\delta_1-1)}\, |x|^{-(2/p+\delta_1)} + C\,\chi_{\{|x|\leq 4\}}.
\end{aligned} \tag{3.10}$$

If $|x| \leq 2$ and $|t| > 2$, then since $\lambda + 1/2 = 1/p + 4\delta_1$, we obtain

$$\begin{aligned}
&|T_{\epsilon,1}^\lambda f(x,t)| + |T_{\epsilon,2}^\lambda f(x,t)| \\
&\leq C\,\chi_{\{|t|\leq 4\}} + C\,\epsilon^{3-2-1/p-2\delta_1} |t|^{-(1/p+2\delta_1)}.
\end{aligned} \tag{3.11}$$

By combining (3.9), (3.10), and (3.11) and using the assumption $\lambda > 1/2$ we obtain

$$\begin{aligned}
&|T_{\epsilon,1}^\lambda f(x,t)| + |T_{\epsilon,2}^\lambda f(x,t)| \\
&\lesssim |x|^{-(\lambda+3/2)}\, |t|^{-1}\, \chi_{\{|x|>2,\,|t|>2,\,4|x|\geq|t|\}} \\
&\quad + |x|^{-3/2}\, |t|^{-(\lambda+1)}\, \chi_{\{|x|>2,\,|t|>2,\,|x|\leq 4|t|\}} \\
&\quad + |x|^{-(2/p+\delta_1)}\, \chi_{\{|x|>2,\,|t|\leq 2\}} + \chi_{\{|x|\leq 4,\,|t|\leq 2\}} \\
&\quad + \chi_{\{|x|\leq 2,\,|t|\leq 4\}} + |t|^{-(1/p+2\delta_1)}\, \chi_{\{|x|\leq 2,\,|t|>2\}}.
\end{aligned} \tag{3.12}$$

Now we consider $T^{\lambda}_{\epsilon,3} f(x,t)$. If $|x| > 2$ and $|t| > 2$, then

$$|T^{\lambda}_{\epsilon,3} f(x,t)| \le C\,\epsilon^3\,(\epsilon|x|)^{-(5/2+1/10)}\,(\epsilon|t|)^{-(3/2-1/10)} \\ \le C\,\epsilon^{3-4}\,|x|^{-(5/2+1/10)}\,|t|^{-(3/2-1/10)}. \tag{3.13}$$

If $|x| > 2$ and $|t| \le 2$, then

$$|T^{\lambda}_{\epsilon,3} f(x,t)| \le C\,\epsilon^3\,(\epsilon|x|)^{-(5/2+1/10)} \int_{\mathbb{R}} \frac{1}{(1+\epsilon|t-s|)^{3/2-1/10}}\,ds \\ \le C\,\epsilon^{3-(5/2+1/10+1)}\,|x|^{-(5/2+1/10)}\,\chi_{\{|t|\le 2\}}. \tag{3.14}$$

If $|x| \le 2$ and $|t| > 2$, then

$$|T^{\lambda}_{\epsilon,3} f(x,t)| \le C\,\epsilon^3\,(\epsilon|t|)^{-(3/2-1/10)} \int_{\mathbb{R}^2} \frac{1}{(1+\epsilon|x-y|)^{5/2+1/10}}\,dy \\ \le C\,\epsilon^{3-(3/2-1/10+2)}\,|t|^{-(3/2-1/10)}\,\chi_{\{|x|\le 2\}}. \tag{3.15}$$

Putting together with (3.13), (3.14), and (3.15), we obtain

$$|T^{\lambda}_{\epsilon,3} f(x,t)| \lesssim |x|^{-(5/2+1/10)}\,\chi_{\{|x|>2,\,|t|\le 2\}} + |t|^{-(3/2-1/10)}\,\chi_{\{|x|\le 2,\,|t|>2\}} \\ + |x|^{-(5/2+1/10)}\,|t|^{-(3/2-1/10)}\,\chi_{\{|x|>2,\,|t|>2\}}. \tag{3.16}$$

If for $c_1, c_2 > 0$, $|x| \le c_1$ and $|t| \le c_2$, then by a change of variables we obtain

$$|T^{\lambda}_{\epsilon,4} f(x,t)| \\ \le C\epsilon^3 \int_{\mathbb{R}}\int_{\mathbb{R}^2} \frac{1}{(1+\epsilon|x-y|)^{2+(\lambda-\frac{1}{2})\frac{1}{2}}(1+\epsilon||t-s|-|x-y||)^{1+(\lambda-\frac{1}{2})\frac{1}{2}}}\,dy\,ds \\ \le C. \tag{3.17}$$

If $|x| \le 2$ and $|t| > 2$, then by the definition of $T^{\lambda}_{\epsilon,4} f(x,t)$, $|t| \le 4$ and by (3.17), $T^{\lambda}_{\epsilon,4} f(x,t)$ is bounded by a constant independent of ϵ. The case $|x| > 2$ and $|t| \le 2$ can be treated in the same way.

If $|x| > 2$, $|t| > 2$, and $||t|-|x|| > 4$, then for any $s \in [-1,1]$ and $y \in [-1,1]^2$, $||t-s|-|x-y|| \ge \frac{1}{2}||t|-|x||$. We therefore obtain

$$|T^{\lambda}_{\epsilon,4} f(x,t)| \\ \le C\epsilon^3 \iint_{Q_0} \frac{1}{(1+\epsilon|x-y|)^{5/2}(1+\epsilon||t-s|-|x-y||)^{\lambda}} \\ \times \chi_{\{2|x-y|\ge|t-s|\ge|x-y|/2\}}\,dy\,ds \\ \le C\epsilon^3\epsilon^{-5/2-\lambda}\frac{1}{|x|^{5/2}||t|-|x||^{\lambda}}\chi_{\{8|x|\ge|t|\ge|x|/8\}} \iint_{Q_0} dy\,ds. \tag{3.18}$$

If $|x| > 2$, $|t| > 2$, and $||t|-|x|| \le 4$, then we take $\delta_p \ge 0$ $(4/5 \le p < 1)$ such that $\delta_p \le 5/2 - 2/p$, $\delta_p < 1/p - 1$, and $\delta_p = 0$ only when $p = 4/5$. Since

$\lambda = 3/p - 5/2$,

$$\begin{aligned} &|T^{\lambda}_{\epsilon,4} f(x,t)| \qquad (3.19)\\ &\le C\epsilon^3 \epsilon^{-2/p-\delta_p} \frac{1}{|x|^{2/p+\delta_p}} \int_{\mathbb{R}} \frac{1}{(1+\epsilon||t-s|-|x-y||)^{1+(1/p-1)-\delta_p}}\, ds \\ &\le C\epsilon^{3-2/p-1-\delta_p} \frac{1}{|x|^{2/p+\delta_p}}. \end{aligned}$$

By combining (3.17), (3.18), and (3.19) we obtain

$$\begin{aligned} |T^{\lambda}_{\epsilon,4} f(x,t)| \lesssim\ & \frac{1}{|x|^{2/p+\delta_p}} \chi_{\{||t|-|x||\le 4,\ |x|>2\}} \\ & + \frac{1}{|x|^{5/2}||t|-|x||^{\lambda}} \chi_{\{9|x|\ge ||t|-|x||\ge 4,\ |x|>2\}} + \chi_{\{|x|\le 4,\ |t|\le 4\}}. \end{aligned} \qquad (3.20)$$

Then (3.12), (3.16), (3.20), and Lemma 3 yield

$$\left|\left\{(x,t)\in\mathbb{R}^3\setminus Q_0^* : \sup_{\epsilon\ge 1} |T^{\lambda}_{\epsilon} f(x,t)| > \alpha/C\right\}\right| \le C\,\alpha^{-p}. \qquad (3.21)$$

Now to consider the complementary case we fix $\epsilon < 1$. This case can be treated in a similar way but we need to control the negative powers of ϵ which occurred during the treatment of the case $\epsilon \ge 1$. These factors are harmless when $\epsilon \ge 1$. When $\epsilon < 1$ it is necessary to gain appropriate positive powers of ϵ without changing the decay of the kernel. This can be done by using moment conditions of atoms. Let P_N be the N-th order Taylor polynomial of the function $s \to \epsilon^3 K(\epsilon|x-y|, \epsilon(t-s))$ expanded about the origin. Then by using the moment conditions on f in Definition 1.(ii) and integrating with respect to s first, we write

$$T^{\lambda}_{\epsilon} f(x,t) = \iint_{Q_0} f(y,s)\,[\epsilon^3\, K(\epsilon|x-y|, \epsilon(t-s)) - P_N(s)]\, ds\, dy.$$

By using the integral version of the mean value theorem, we obtain

$$\begin{aligned} &\left|\epsilon^3 K(\epsilon|x-y|, \epsilon(t-s)) - P_N(s)\right| \\ &\lesssim \epsilon^3 \int_{[0,1]^{N+1}} \left|\frac{\partial^{N+1}}{\partial s^{N+1}}\left[K(\epsilon|x-y|, \epsilon(t-\hat{u}s))\right]\right| du_1\cdots du_{N+1} \\ &\lesssim \epsilon^3 \epsilon^{(N+1)} \int_{[0,1]^{N+1}} \left|\frac{\partial^{N+1} K}{\partial s^{N+1}}(\epsilon|x-y|, \epsilon(t-\hat{u}s))\right| du_1\cdots du_{N+1} \end{aligned}$$

where $\hat{u} = \prod_{i=1}^{N+1} u_i$. Since we gain $\epsilon^{(N+1)}$ and the kernel K has the same decay after taking derivatives, we use the same argument as above to obtain (3.12), (3.16), and (3.20) for $\epsilon < 1$, which imply

$$\left|\left\{(x,t)\in\mathbb{R}^3\setminus Q_0^* : \sup_{\epsilon< 1} |T^{\lambda}_{\epsilon} f(x,t)| \ > \ \alpha/C\right\}\right| \le C\,\alpha^{-p}. \qquad (3.22)$$

One has to notice that for $\epsilon \geq 1$ we use a change of variables to obtain negative power of ϵ in (3.16), (3.17), and (3.19) but for $\epsilon < 1$ one can use the fact that the integrands are bounded above by 1. We use (3.7), (3.21), and (3.22) to prove (3.6) for the cube Q_0.

Now we suppose that f is a (p, N)-atom ($N \geq 3(1/p - 1)$), supported in a cube Q of diameter δ centered at (x_Q, t_Q). By translation invariance we can assume $(x_Q, t_Q) = (0, 0)$. Let $h(x, t) = \delta^{3/p} f(\delta x, \delta t)$. Then h is an atom supported in the cube Q_0 centered at (0, 0) and we write

$$
\begin{aligned}
T_\epsilon^\lambda f(x,t) &= \int_{\mathbb{R}}\int_{\mathbb{R}^2} \delta^{-3/p}\, h(\delta^{-1}(x-y), \delta^{-1}(t-s))\, G^\epsilon(y,s)\, dy\, ds \\
&= \delta^{-3/p}\, T_{\epsilon\delta}^\lambda h(\delta^{-1}x, \delta^{-1}t),
\end{aligned}
$$

which implies

$$
\sup_{\epsilon>0} |T_\epsilon^\lambda f(x,t)| = \delta^{-3/p} \sup_{\epsilon>0} |T_\epsilon^\lambda h(\delta^{-1}x, \delta^{-1}t)|.
$$

We therefore have

$$
\begin{aligned}
&\left|\left\{(x,t) \in \mathbb{R}^3 \,:\, T_*^\lambda f(x,t) \;>\; \alpha/C\right\}\right| \\
&= \left|\left\{(x,t) \in \mathbb{R}^3 \,:\, T_*^\lambda h(\delta^{-1}x, \delta^{-1}t) \;>\; \delta^{3/p}\alpha/C\right\}\right| \\
&\leq C(\delta^{3/p}\alpha)^{-p}\delta^3 = C\alpha^{-p}.
\end{aligned}
$$

This completes the proof. □

We are now ready to prove Theorem 2 which is an immediate consequence of Proposition 1 and Lemma 2.

Proof of Theorem 2 Let $f = \sum_{j=1}^\infty c_j f_j \in H^p(\mathbb{R}^3)$ where f_j's are (p, N)-atoms and $\sum_{j=1}^\infty |c_j|^p < \infty$. Due to the integrability of the kernel (see Lemma 1 and also [2]) and ℓ^1 convergence of the coefficient $\{c_j\}$, it is easy to see that $T_\epsilon^\lambda f$ is well defined and can be written

$$
T_\epsilon^\lambda f = \sum_{j=1}^\infty c_j T_\epsilon^\lambda f_j.
$$

We therefore have

$$
T_*^\lambda f(x,t) = \sup_{\epsilon>0} |T_\epsilon^\lambda f(x,t)| \leq \sum_{j=1}^\infty |c_j| \sup_{\epsilon>0} |T_\epsilon^\lambda f_j(x,t)| = \sum_{j=1}^\infty |c_j| T_*^\lambda f_j(x,t).
$$

By Proposition 1, we have

$$
|\{(x,t) \,:\, T_*^\lambda f_j(x,t) \geq \alpha\}| \leq C\alpha^{-p}.
$$

Now we apply Lemma 2 to obtain

$$|\{(x,t)\,:\,T_*^\lambda f(x,t) \geq \alpha\}| \leq C\alpha^{-p} \sum_{j=1}^{\infty} |c_j|^p.$$

By taking the infimum on the right-hand side we obtain the desired estimates. □

We turn to the proof of Theorem 1. If λ is the critical index, that is, $\lambda = 3/p - 5/2$ we used (3.12) (3.16), and (3.20) to obtain weak type estimates but the decay of the kernel was not large enough to obtain strong type estimates. At this moment it is worth recalling that (3.12), (3.16), and (3.20) are valid for all $\epsilon > 0$. To prove Theorem 1 we basically use the same estimates and we also take advantage of our assumption that λ is greater than the critical index and p is not the extreme exponent, that is, $4/5 < p < 1$.

Proof of Theorem 1 Let f be a (p, N)-atom $(N \geq 3(1/p - 1))$ on $\mathbb{R}^3$ and supported in a cube Q. Due to the translation invariance and maximality of T_*^λ as above we may assume that Q is centered at the origin with diameter 1. In view of atomic decomposition, it suffices to show that there exists a C independent of f such that

$$|| \sup_{\epsilon>0} |T_\epsilon^\lambda f| \,||_{L^p(\mathbb{R}^3)} \leq C.$$

By the integrability of the kernel, we have $|T_\epsilon^\lambda f(x)| \leq C||K||_{L^1}|Q|^{-1/p}$ and thus

$$|| \sup_{\epsilon>0} |T_\epsilon^\lambda f| \,||^p_{L^p(Q^*)} \leq C \iint_{Q^*} |Q|^{-\frac{1}{p}\cdot p}\, dx\, dt \leq C.$$

For the complementary case we use the notation in (3.8) and the pointwise estimates in (3.12), (3.16), and (3.20) to write

$$\begin{aligned}
&||T_*^\lambda f||^p_{L^p(\mathbb{R}^3\setminus Q^*)} \\
&\quad = \iint_{\mathbb{R}^3\setminus Q^*} |T_*^\lambda f(x,t)|^p\, dx\, dt \\
&\quad \leq \iint_{\mathbb{R}^3\setminus Q^*} \Big[\sup_{\epsilon>0}(|T_{\epsilon,1}^\lambda f(x,t)| + |T_{\epsilon,2}^\lambda f(x,t)| + |T_{\epsilon,3}^\lambda f(x,t)| \\
&\qquad + |T_{\epsilon,4}^\lambda f(x,t)|)\Big]^p\, dx\, dt \\
&\quad \lesssim \iint_{\mathbb{R}^3} |x|^{-(\lambda+3/2)p}\, |t|^{-p}\, \chi_{\{|x|>2,\ |t|>2,\ 4|x|\geq|t|\}} \\
&\qquad + |x|^{-3p/2}\, |t|^{-(\lambda+1)p}\, \chi_{\{|x|>2,\ |t|>2,\ |x|\leq 4|t|\}} \\
&\qquad + |x|^{-(2/p+\delta_1)p}\, \chi_{\{|x|>2,\ |t|\leq 4\}} \\
&\qquad + |t|^{-(1/p+2\delta_1)p}\, \chi_{\{|x|\leq 4,\ |t|>2\}} + \chi_{\{|x|\leq 2,\ |t|\leq 4\}}
\end{aligned}$$

$$
\begin{aligned}
&+ |x|^{-(5/2+1/10)p}\, |t|^{-(3/2-1/10)p}\, \chi_{\{|x|>2,\ |t|>2\}} \\
&+ |x|^{-(5/2+1/10)p}\, \chi_{\{|x|>2,\ |t|\le 2\}} \\
&+ |t|^{-(3/2-1/10)p}\, \chi_{\{|x|\le 2,\ |t|>2\}} \\
&+ |x|^{-(2/p+\delta_p)} \chi_{\{|x|>2,\ ||t|-|x||\le 4\}} \\
&+ |x|^{-5p/2} ||t|-|x||^{-\lambda p} \chi_{\{|x|>2,\ 9|x|\ge ||t|-|x||\ge 4\}}\, dx\, dt.
\end{aligned}
$$

We set $2\delta_0 = \lambda - 3/p + 5/2 > 0$. We use $p < 1$ to obtain

$$
\begin{aligned}
&\iint_{\mathbb{R}^3} |x|^{-(\lambda+3/2)p}\, |t|^{-p}\, \chi_{\{|x|>2,\ |t|>2,\ 4|x|\ge |t|\}}\, dx\, dt \\
&\quad + \iint_{\mathbb{R}^3} |x|^{-3p/2}\, |t|^{-(\lambda+1)p}\, \chi_{\{|x|>2,\ |t|>2,\ |x|\le 4|t|\}}\, dx\, dt \\
&\lesssim \iint_{\mathbb{R}^3} |x|^{-(2/p+\delta_0)p}\, |t|^{-(1/p+\delta_0)p}\, \chi_{\{|x|>2,\ |t|>2\}}\, dx\, dt \;\le\; C,
\end{aligned}
$$

and use $\delta_1 > 0$ to obtain

$$
\iint_{\mathbb{R}^3} \left[\, |x|^{-(2/p+\delta_1)p}\, \chi_{\{|x|>2,\ |t|\le 4\}} + |t|^{-(1/p+2\delta_1)p}\, \chi_{\{|x|\le 4,\ |t|>2\}} \right] dx\, dt \le C.
$$

For the fifth through the seventh integral, from $p > 4/5$ we see that $(5/2 + 1/10)p > 2$ and $(3/2 - 1/10)p > 1$. Thus, we get

$$
\begin{aligned}
&\iint_{\mathbb{R}^3} \left[\, |x|^{-(5/2+1/10)p}\, |t|^{-(3/2-1/10)p}\, \chi_{\{|x|>2,\ |t|>2\}} \right.\\
&\quad \left. + |x|^{-(5/2+1/10)p} \chi_{\{|x|>2,\ |t|\le 2\}} \right] dx\, dt \\
&\quad + \iint_{\mathbb{R}^3} |t|^{-(3/2-1/10)p}\, \chi_{\{|x|\le 2,\ |t|>2\}}\, dx\, dt \;\le\; C.
\end{aligned}
$$

Finally, since $\delta_p > 0$ when $p > 4/5$, we obtain

$$
\begin{aligned}
&\iint_{\mathbb{R}^3} \left[\, |x|^{-(2/p+\delta_p)p} \chi_{\{|x|>2,\ ||t|-|x||\le 4\}} \right.\\
&\quad \left. + |x|^{-5p/2} ||t|-|x||^{-\lambda p} \chi_{\{|x|>2,\ 9|x|\ge ||t|-|x||\ge 4\}} \right] dx\, dt \\
&\quad \lesssim \iint_{\mathbb{R}^3} \left[\, |x|^{-(2/p+\delta_p)p}\, \chi_{\{|x|>2,\ |t'|\le 4\}} \right.\\
&\quad \left. + |x|^{-5p/2}\, |t'|^{-\lambda p}\, \chi_{\{|x|>2,\ 9|x|\ge |t'|\ge 4\}} \right] dx\, dt' \;\le\; C.
\end{aligned}
$$

This completes the proof. □

4 Sharpness

In this section we shall show that the estimates in the theorems are sharp in the sense that if $\lambda \le \lambda_p$ or $p \le 4/5$ then even the convolution operator T^λ is unbounded on L^p. To do this we take a compactly supported smooth function ψ which is identically equal to 1 in $\{(\xi', \xi_3) \mid \max\{|\xi'|, |\xi_{d+1}|\} \le 1\}$. We then have $T^\lambda(F^{-1}\psi) = K_\lambda$ where F^{-1} is the inverse Fourier transform and K_λ is the kernel of T^λ. Therefore it suffices to show that if $\lambda \le \lambda_p$ or $p \le 4/5$ then $||K_\lambda||_p^p = +\infty$. To do this we first obtain an asymptotic behavior of K_λ when $r > 2$ and $t > 2$. It was obtained by H. Luer in [2] by using asymptotic behavior of the Bessel function. To make the paper self-contained we briefly describe Luer's idea for the region we use. Throughout this section we write $f(v) \approx v^\alpha$ as $v \to \infty$ if there exist $C > 0$ and $D > 0$ such that for $|v| \ge D$, $f(v)$ has the same sign and $C^{-1}v^\alpha \le |f(v)| \le Cv^\alpha$.

We write

$$
\begin{aligned}
K_\lambda(r,t) &= C_\lambda \int_0^1 (1-s)^{\lambda-1} s^3 (rs)^{-1} J_1(rs)(ts)^{-1/2} J_{1/2}(ts)ds \\
&= C_\lambda \int_0^1 (1-s)^{\lambda-1} s^3 (rs)^{-1} J_1(rs) \frac{\sin st}{st} ds \\
&= C_\lambda \int_0^{1/r} + \int_{1/r}^1 (1-s)^{\lambda-1} s^3 (rs)^{-1} J_1(rs) \frac{\sin st}{st} ds \\
&= I_1 + I_2.
\end{aligned}
$$

By using a change of variables we obtain

$$
\begin{aligned}
|I_1| &= C_\lambda \left| \int_0^{1/r} (1-s)^{\lambda-1} s^3 (rs)^{-1} J_1(rs) \frac{\sin st}{st} ds \right| \\
&= \frac{C_\lambda}{r^3 t} \left| \int_0^1 (1-\frac{u}{r})^{\lambda-1} u J_1(u) \sin(ut/r) du \right| \\
&\le \frac{C}{r^3 t}.
\end{aligned}
$$

For I_2 we use the asymptotic expansion of the Bessel functions to obtain

$$
\begin{aligned}
I_2 &= C \int_{1/r}^1 (1-s)^{\lambda-1} s^3 E(rs) \frac{\sin st}{st} ds \\
&\quad + C \int_{1/r}^1 (1-s)^{\lambda-1} s^3 (rs)^{-3/2} \cos\left(rs - \frac{\pi}{4}\right) \frac{\sin st}{st} ds \\
&= I_3 + I_4
\end{aligned}
$$

where $E(v) \approx v^{-5/2}$ as $v \to \infty$. It is easy to see that

$$
|I_3| \le \frac{C}{r^{5/2} t}.
$$

To treat I_4 we write

$$\begin{aligned} I_4 &= \int_0^1 - \int_0^{1/r} (1-s)^{\lambda-1} s^3 (rs)^{-3/2} \cos\left(rs - \frac{\pi}{4}\right) \frac{\sin st}{st} ds \\ &= I_5 - I_6. \end{aligned}$$

For I_6 we obtain

$$|I_6| \leq \frac{C}{r^3 t}.$$

Now it remains to treat I_5. To do this we use an elementary identity of trigonometric functions to obtain

$$2\cos\left(rs - \frac{\pi}{4}\right) \sin st = \sin\left[(r+t)s - \pi/4\right] - \sin\left[(r-t)s - \pi/4\right].$$

We plug this into I_5 to obtain

$$\begin{aligned} I_5 &= \frac{C}{r^{3/2} t} \int_0^1 (1-s)^{\lambda-1} s^{1/2} \left(\sin\left[(r+t)s - \pi/4\right] - \sin\left[(r-t)s - \pi/4\right]\right) ds \quad (4.1) \\ &= I_6 + I_7. \end{aligned}$$

We can write

$$I_5 = \frac{C}{r^{3/2}} \int_{-1}^1 \int_0^1 (1-s)^{\lambda-1} s^{3/2} \cos\left[(r+zt)s - \pi/4\right] ds dz. \qquad (4.2)$$

If $|r-t| \leq 1/4$, then the second integrand in (4.1) has the constant sign in the domain of the integral so $I_7 \approx r^{-3/2} t^{-1}$ as $r \to \infty$. Furthermore $I_6 \approx r^{-5/2} t^{-1} (r+t)^{-1}$ as $r \to \infty$. Therefore if $|r-t| \leq 1/4$, then $I_5 \approx r^{-3/2} t^{-1}$ as $r \to \infty$. If $r \geq 4t$, then by (4.2) $I_5 \approx C\, r^{-3/2-\lambda}$ as $r \to \infty$.

If $\lambda < 3/p - 5/2$ and $0 < p < 1$, then $(-3/2 - \lambda)p + 1 > -2$ so

$$\begin{aligned} ||K_\lambda||_p^p &\geq \int \int_{t \geq 2,\ r \geq 4t} |K_\lambda(r,t)|^p\, r\, dr\, dt \\ &\geq \int_2^\infty \int_{4t}^\infty r^{(-3/2-\lambda)p}\, r\, dr\, dt \\ &= +\infty. \end{aligned}$$

If $p \leq 4/5$, then $-5p/2 + 1 \geq -1$ so

$$\begin{aligned} ||K_\lambda||_p^p &\geq \int \int_{r \geq 2,\ |r-t| \leq 1/4} |K_\lambda(r,t)|^p\, r\, dr\, dt \\ &\geq \int_2^\infty \int_{r-1/4}^{r+1/4} r^{-5p/2+1}\, dt\, dr \\ &= 1/2 \int_2^\infty r^{-5p/2+1}\, dr \\ &= +\infty. \end{aligned}$$

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