BOCHNER-RIESZ MEANS WITH RESPECT TO A 2 BY 2 CYLINDER

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Abstract. The generalized Bochner-Riesz operator $S^{R,\lambda}$ may be defined as

$$S^{R,\lambda}f(x) = \mathcal{F}^{-1}\left[\left(1-\frac{\rho}{R}\right)_+^\lambda \widehat{f}\right](x)$$

where ρ is an appropriate distance function and $\mathcal{F}^{-1}$ is the inverse Fourier transform. The sharp bound $\|S^{R,\lambda}f\|_{L^4(\mathbf{R}^2\times\mathbf{R}^2)} \leq C\|f\|_{L^4(\mathbf{R}^2\times\mathbf{R}^2)}$ is shown for the distance function $\rho(\xi',\xi") = \max\{|\xi'|,|\xi''|\}$. This is a rough distance function corresponding to the $\mathbf{R}^4$ cylinder analog $\{(x_1,x_2,x_3,x_4)\in\mathbf{R}^4, \quad x_1^2+x_2^2\leq 1, x_3^2+x_4^2\leq 1\}$.

Introduction

Bochner-Riesz means were conceived as a method for addressing the convergence of the inverse Fourier transform. We know the inverse Fourier transform (denoted $\mathcal{F}^{-1}$) will allow us to recover f from the Fourier transform of f (denoted $\widehat{f}$) if f is in Schwartz space (smooth, rapidly decaying at ∞, denoted $\mathcal{S}$) or if $f\in L^2$. However, for more general $f\in L^p$, $\widehat{f}$ will typically not be integrable and the inverse Fourier transform will not represent a convergent Lebesgue integral. Bochner-Riesz means allow one to check the convergence of the integral as a limit.

We define distance functions to be functions ρ which are continuous on $\mathbf{R}^d$ and satisfy

$$\begin{aligned}\rho(tx) &= t\rho(x), \qquad t>0,\\ \rho(x) &> 0 \qquad \text{if } x\neq 0.\end{aligned}$$

We define the generalized Bochner-Riesz operator $S^{R,\lambda}$ as follows:

$$S^{R,\lambda}f(x) = \mathcal{F}^{-1}[\left(1-\frac{\rho}{R}\right)_+^\lambda \widehat{f}](x). \tag{0.1}$$

Here $(g(\xi))_+ = \max\{g(\xi),0\}$ is the positive part. Note that as $R\to\infty$, $S^{R,\lambda}f\to f$ for $f\in\mathcal{S}$. Also note that $S^{R,\lambda}f\to f$ for $f\in L^2$ by Plancherel's theorem. On general principals, the question of convergence on other L^p spaces is equivalent to the question of boundedness of the operators $S^{R,\lambda}$. By scaling, we may also assume that $R=1$. From now on we will focus on the boundedness of $S^\lambda = S^{1,\lambda}$ on L^p.

Standard Bochner-Riesz means, where $\rho(\xi)=|\xi|$, have been studied extensively. This case will be referred to as spherical means, as the multiplier is supported on a spherical ball. In 1971, Fefferman [2] showed that for spherical means to be

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Date: July 21, 2008.
2000 *Mathematics Subject Classification.* Primary 42B15; Secondary 42B25.
Key words and phrases. Fourier Analysis, Multipliers, Bochner-Riesz Means, Cone Multiplier.

bounded on L^p, it is necessary to have $\lambda > \lambda^*(p) = \max\{d\left|\frac{1}{2} - \frac{1}{p}\right| - \frac{1}{2}, 0\}$. The Bochner-Riesz conjecture states that this is both necessary and sufficient.

In 1972, Carleson and Sjölin verified the conjecture in $\mathbf{R}^2$ [1]. In 1973, Fefferman showed a connection between the Restriction Conjecture for the sphere and the Bochner-Riesz conjecture [3]. Recent progress on the Restriction Conjecture has used a bilinear approach (as in [8]) and recently Lee adapted Fefferman's argument to apply the bilinear results directly to spherical means [4]. This approach applied to the recent bilinear result of Tao [7] proves the Bochner-Riesz conjecture for $p > 2 + \frac{4}{d}$ for $d \geq 3$, the best current range of p.

To gain some insight into general ρ we consider $\rho(\xi) = \max\{|\xi'|, |\xi''|\}$, a distance function related to a cylinder. Here $\xi = (\xi', \xi'') \in \mathbf{R}^{d'} \times \mathbf{R}^{d''}$ and we assume $d' \geq d''$. For this ρ, we denote the multiplier by m_λ:

$$m_\lambda(\xi) = (1 - \max\{|\xi'|, |\xi''|\})^\lambda_+. \tag{0.2}$$

The support of m_λ is a cylinder analog where d'' is allowed to be bigger than 1. For ξ near $|\xi'| = 1$, $m_\lambda = (1 - |\xi'|)^\lambda_+$ which is the multiplier for spherical means in $\mathbf{R}^{d'}$. A similar relationship holds between ξ and ξ'' for ξ near $|\xi''| = 1$. This way the same critical index $\lambda^*(p)$ will apply for the cylinder multiplier and spherical means in $\mathbf{R}^{d_1}$, since we assume $d' \geq d''$ and therefore will provide the more restrictive condition. Given these observations, one might expect that this cylindrical operator would be bounded for the same range of λ as spherical means in $\mathbf{R}^{d'}$. However, this is not the case.

This problem was first studied by Luers [5] in 1988. Along with some partial positive results, he showed that in $\mathbf{R}^d \times \mathbf{R}$ when $d \geq 4$ and $\lambda^*(p) \geq 1$, then S^λ is unbounded for all λ. This is curious, since with spherical means there is always some large λ for which the operator is bounded.

In 2007 the author made a more thorough investigation of the $\mathbf{R}^d \times \mathbf{R}$ case [9]. It was shown that since the multiplier is not smooth on the light cone $|\xi'| = |\xi''|$, boundedness would also depend on the cone multiplier. The nature of the non-smoothness requires that $\lambda^*(p)$, which is also the critical index the cone multiplier in question, must be less than 1, rather than simply less than λ.

This paper addresses a case not treated in [9], the case where $\xi \in \mathbf{R}^2 \times \mathbf{R}^2$. The arguments in [9] made ample use of the simplifying assumption that $d'' = 1$, and having $d'' > 1$ greatly complicates matters. Large portions of the argument are easily adapted for use here, however in the key region where $|\xi'| \approx |\xi''|$ more significant modifications become necessary.

1. Results

Theorem 1.1. *The operator S_λ is bounded from $L^4(\mathbf{R}^2 \times \mathbf{R}^2) \to L^4(\mathbf{R}^2 \times \mathbf{R}^2)$ for $\lambda > 0$.*

Our operator has quantitatively similarities to spherical means in $\mathbf{R}^2$, so they will share the critical index of $\lambda > 0$ when $p = 4$. However, our operator also shares characteristics with the cone-like multiplier supported near $|\xi'| = |\xi''|$. Sharp results are not available for this multiplier, so no sharp results for $p > 4$ were obtained here. However the cone region can be sliced into numerous neighborhoods of $S^1 \times S^1$, where S^1 represents a sphere in $\mathbf{R}^2$ (that is, a circle). Since the critical index for spherical means in $\mathbf{R}^2$ is $\lambda > 0$ when $p = 4$, adding these slices results in no significant loss. To extend the result to $p > 4$, some kind of sharp result for the $|\xi'| = |\xi''|$ multiplier will be required.

2. Proof

We begin by defining a new operator, S_δ, related to spherical means.

$$\widehat{S_\delta f}(\xi) \quad = \quad \varphi(\delta^{-1}(1 - |\xi|))\hat{f}(\xi). \tag{2.1}$$

By using a dyadic decomposition, it is easy to see that the study of spherical means can by reduced to the study of this operator. φ is in the class Φ, which consists of all function in $\mathcal{C}_0^\infty$ with the following properties:

$$\begin{aligned} \text{support of } \varphi \quad &\subset \quad [0,2], \\ \left|\frac{\partial^{|\alpha|}}{\partial \xi^\alpha}\varphi\right| \quad &\lesssim \quad C \qquad \text{for all } |\alpha| \le d+2, \end{aligned}$$

where $\alpha = (\alpha_1, \ldots, \alpha_{d+1})$ is a standard multi-index and C is a fixed constant. κ is a $\mathcal{C}_0^\infty$ function supported in $(-\frac{1}{2}, \frac{1}{2})$. δ is assumed to be a small positive number. As stated in the introduction, the sharp bounds for S_δ were verified by Carleson and Sjölin.

Theorem 2.1 (Carleson, Sjölin)**.**

$$\|S_\delta g\|_{L^4(\mathbf{R}^2)} \lesssim \delta^{-\lambda}\|g\|_{L^4(\mathbf{R}^2)},$$

for all $\lambda > 0$.

Note that throughout the exposition, we will be using the symbol $\lesssim$ to denote that one expression is less than the other up to a constant which depends only on fixed values such as dimension and constants ϵ which are fixed at the beginning of proofs.

We now make a preliminary decomposition, then group the pieces into four cases which capture the nature of the multiplier $m_\lambda = (1 - \max\{|\xi'|, |\xi''|\})_+^\lambda$. Choose $\varphi \in \mathcal{C}_0^\infty(\mathbf{R})$ so that the support of φ is contained in $[\frac{1}{2}, 2]$, $\varphi \neq 0$ on $[\frac{3}{4}, \frac{3}{2}]$ and $\sum_{j=2}^\infty \varphi(2^j t) \equiv 1$ for $t \in (0, \frac{1}{4})$. Choose φ_1 so that $\varphi_1(t) + \sum_{j=2}^\infty \varphi(2^j(1-t)) \equiv 1$ for $t \in [0,1)$.

Let

$$\begin{aligned} m_{jk}(\xi) \quad &= \quad \varphi(2^j(1-|\xi'|))\varphi(2^k(1-|\xi''|))m_\lambda(\xi) \qquad \text{for } j,k \ge 2, \\ m_{1k}(\xi) \quad &= \quad \varphi_1(|\xi'|)\varphi(2^k(1-|\xi''|))m_\lambda(\xi) \qquad \text{for} \quad k \ge 2, \\ m_{j1}(\xi) \quad &= \quad \varphi(2^j(1-|\xi'|))\varphi_1(|\xi''|)m_\lambda(\xi) \qquad \text{for} \quad j \ge 2, \\ m_{11}(\xi) \quad &= \quad \varphi_1(|\xi'|)\varphi_1(|\xi''|)m_\lambda(\xi). \end{aligned} \tag{2.2}$$

Then, using the Triangle inequality,

$$\|m_\lambda\|_{M^p} \le \sum_{k,j=1}^\infty \|m_{jk}(\xi)\|_{M^p}, \tag{2.3}$$

where $\|m_\lambda\|_{M^p}$ denotes the operator norm of $f \to \mathcal{F}^{-1}[m_\lambda \widehat{f}]$.

We divide the sum into three parts and deal with each of them separately. The three parts are

Case (i) $|k-j| \geq 2$: When $|k-j| \geq 2$ the support of m_{jk} is near the "sides" of the cylinder, where $|\xi'| = 1$ or $|\xi''| = 1$. In this region $m_\lambda = (1-|\xi'|)^\lambda$ (or $m_\lambda = (1-|\xi''|)^\lambda$), so one expects the multiplier to behave like spherical means in $\mathbf{R}^2$.

Case (ii) $|k-j| \leq 1$, $k, j \neq 1$: This region contains the set where $|\xi'| = |\xi_{d+1}|$ (except near the origin). This is the most interesting case, where we explore the non-smooth portion of m_λ.

Case (iii) $|k-j| \leq 1$, $k = 1$ or $j = 1$: We simply use scaling to extend our results from the first two cases to the center of the cylinder.

2.1. **Case (i) $|k-j| \geq 2$, near $|\xi'| = 1$ or $|\xi''| = 1$.**

Lemma 2.2. *Let $\lambda > 0$. The following sums converge:*

$$\sum_{k=1}^\infty \sum_{j=k+2}^\infty \|m_{jk}\|_{M^4}, \tag{2.4}$$

$$\sum_{j=1}^\infty \sum_{k=j+2}^\infty \|m_{jk}\|_{M^4}. \tag{2.5}$$

These sums represent all multiplier pieces where $|k-j| \geq 2$.

Proof. Fix a small $\epsilon > 0$.

Consider the first sum. When $j \geq k+2 \geq 4$, we have $\rho(\xi) = |\xi'|$, so

$$m_{jk}(\xi) = \varphi(2^j(1-|\xi'|))\varphi(2^k(1-|\xi''|))(1-|\xi'|)^\lambda.$$

Note that

$$\begin{aligned} \|m_{jk}\|_{M_4} &= \|S'_{2^{-j}}\|_{L^4(\mathbf{R}^2)\to L^4(\mathbf{R}^2)} \|\widetilde{S}_{2^{-k}}\|_{L^4(\mathbf{R}^2)\to L^4(\mathbf{R}^2)}, \\ \text{where} \quad S'_{2^{-j}} f &= \mathcal{F}^{-1}[\varphi(2^j(1-|\xi'|))(1-|\xi'|)^\lambda] * f, \\ \text{and} \quad \widetilde{S}_{2^{-k}} f &= \mathcal{F}^{-1}[\varphi(2^k(1-|\xi''|))] * f. \end{aligned}$$

We note that for $S'_{2^{-j}}$ we can use Theorem 2.1 with $\delta = 2^{-j}$ as follows. Define $\widetilde{\varphi}(t) = t^\lambda \varphi(t)$. Then

$$S'_{2^{-j}} f = 2^{-j\lambda} \mathcal{F}^{-1}[\widetilde{\varphi}(2^j(1-|\xi'|))] * f. \tag{2.6}$$

Since Theorem 2.1 requires only that φ a C_0^∞ function, we obtain the following resulting bound. Here we are also using the fact that since the operator $S'_{2^{-j}}$ acts only on the first d variables, leaving the ξ'' variables independent, we need only investigate the corresponding operator on $\mathbf{R}^2$, for which we use the same name.

$$\|S'_{2^{-j}}\|_{L^4(\mathbf{R}^2)\to L^4(\mathbf{R}^2)} \lesssim 2^{-j\lambda} 2^{j\epsilon}. \tag{2.7}$$

The operator $\widetilde{S}_{2^{-k}}$ only acts on the ξ'' variables, so we consider the corresponding operator on $\mathbf{R}^2$. Here Theorem 2.1 also applies, this time without changing φ. We obtain the following bound:

$$\|\widetilde{S}_{2^{-k}}\|_{L^4(\mathbf{R}^2)\to L^4(\mathbf{R}^2)} \lesssim 2^{k\epsilon}. \tag{2.8}$$

The same bounds for j and k such that $j \geq k+2 = 3$ can be derived in a similar way. Now we can sum over the range $j \geq k+2$.

$$\sum_{k=1}^\infty \sum_{j=k+2}^\infty \|m_{jk}\|_{M^4} \lesssim \sum_{k=1}^\infty \sum_{j=k+2}^\infty 2^{j(-\lambda+\epsilon)} 2^{k\epsilon}. \tag{2.9}$$

This sum converges if $\lambda > 2\epsilon$.

The second sum where $k \geq j+2$ converges for by a similar argument. ∎

2.2. **Case (ii)** $|k-j| \le 1, k, j \ne 1$**, near** $|\xi'| = |\xi''|$**.** In this region we investigate the non-smooth portion of the multiplier. A number of steps are taken to prepare the multiplier, first to highlight the relationship to the cone then to decompose the multiplier into appropriate pieces.

2.2.1. *Preparation of the Multiplier.* We begin by subtracting a smooth multiplier in order to make our multiplier zero on the set where $|\xi'| = |\xi''|$. This will simplify our analysis. We then decompose the multiplier dyadically with respect to distance from $|\xi'| = |\xi''|$. The multiplier will have approximately uniform magnitude on the support of each of these dyadic pieces. A further decomposition essentially leaves us with pieces which are supported on a neighborhood of $S^1 \times S^1$. We will then be able to apply the Bochner-Riesz results from $\mathbf{R}^2$ to these final pieces.

Fix j, k such that $|j-k| \le 1$ and $j, k \ne 1$. For the multiplier piece m_{jk}, we set it equal to zero along $|\xi'| = |\xi''|$ as follows. Let

$$\begin{aligned}\mu_{jk}(\xi) &= m_{jk}(\xi) - \varphi(2^j(1-|\xi'|))\varphi(2^k(1-|\xi''|))(1-|\xi'|)^\lambda \\ &= \begin{cases} \varphi(2^j(1-|\xi'|))\varphi(2^k(1-|\xi''|))[(1-|\xi''|)^\lambda - (1-|\xi'|)^\lambda] & \text{if } |\xi''| \ge |\xi'|, \\ 0 & \text{if } |\xi''| \le |\xi'|. \end{cases}\end{aligned} \tag{2.10}$$

Referring back to the proof of Lemma 2.2, we see that the multiplier we are subtracting,

$$\varphi(2^j(1-|\xi'|))\varphi(2^k(1-|\xi''|))(1-|\xi'|)^\lambda,$$

is nearly the same as the multipliers considered in Case (i). The only difference is that $|k-j| \le 1$, whereas in Case (i) $j \ge k+2$. The same proof applies here, so

$$\|\varphi(2^j(1-|(\cdot)'|))\varphi(2^k(1-|(\cdot)''|))(1-|(\cdot)'|)^\lambda\|_{M^p} \lesssim 2^{j(-\lambda+\lambda^*(p)+2\epsilon)}.$$

We can sum these terms for $p = 4$ and $\{j, k, \quad |k-j| \le 1, j \ge 2\}$ as long as $\lambda > 2\epsilon$.

Case (ii) is now reduced to finding a bound for $\|\mu_{jk}\|_{M_p}$.

Decompose the support of μ_{jk} in the following manner. Let $\phi \in \mathcal{C}_0^\infty(\mathbf{R})$ be a function such that $\phi \equiv 1$ on $[\frac{1}{4}, 4]$ and $supp \phi \subset [\frac{1}{8}, 8]$. Let $\widetilde{\varphi}(t) = \varphi(t)t$, $\widetilde{\phi}(t) = \phi(t)t^{\lambda-1}$ and $h(\xi) = 1 - \frac{|\xi'|}{|\xi''|}$. Then

$$\begin{aligned}(1-|\xi''|)^\lambda - (1-|\xi'|)^\lambda &= \int_0^1 -\lambda[1-(t|\xi''|+(1-t)|\xi'|)]^{\lambda-1}[|\xi''|-|\xi'|]dt \\ &= -\lambda|\xi''|\left|1-\frac{|\xi'|}{|\xi''|}\right|\int_0^1 [1-(t|\xi''|+(1-t)|\xi'|)]^{\lambda-1}dt,\end{aligned} \tag{2.11}$$

$$
\begin{aligned}
\varphi(2^M h(\xi))\mu_{jk}(\xi) &= \varphi(2^M h(\xi))\varphi(2^j(1-|\xi'|)) \\
&\quad \cdot\ \varphi(2^k(1-|\xi''|))[(1-|\xi''|)^\lambda - (1-|\xi'|)^\lambda] \\
&= \varphi(2^M h(\xi))\varphi(2^j(1-|\xi'|))\varphi(2^k(1-|\xi''|)) \qquad (2.12)\\
&\quad \cdot\ \int_0^1 (\phi(2^j[1-(t|\xi''|+(1-t)|\xi'|)])(-\lambda)|\xi''| \\
&\quad \cdot\ \left|1-\frac{|\xi'|}{|\xi''|}\right| [1-(t|\xi''|+(1-t)|\xi'|)]^{\lambda-1})dt \\
&= (-\lambda)|\xi''|2^{-M}2^{-j(\lambda-1)}\widetilde{\varphi}\left(2^M h(\xi)\right)\varphi(2^j(1-|\xi'|)) \qquad (2.13)\\
&\quad \cdot\ \varphi(2^k(1-|\xi''|))\int_0^1 \widetilde{\phi}(2^j[1-(t|\xi''|+(1-t)|\xi'|)])dt \\
&= (-\lambda)|\xi''|2^{-M}2^{-j(\lambda-1)}\widetilde{\varphi}\left(2^M h(\xi)\right)\varphi_j(\xi')\varphi_k(\xi'')\int_0^1 \phi_{j,t}(\xi)dt, \qquad (2.14)
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi_j(\xi') &= \varphi(2^j(1-|\xi'|)), \qquad (2.15)\\
\varphi_k(\xi'') &= \varphi(2^k(1-|\xi''|)), \qquad (2.16)\\
\phi_{j,t}(\xi) &= \widetilde{\phi}(1-(t|\xi''|+(1-t)|\xi'|)). \qquad (2.17)
\end{aligned}
$$

Since $|k-j|\le 1$, $\phi_{j,t}$ satisfies the following properties:

$$
\begin{aligned}
supp(\phi_{j,t}) &\subset \{\xi\in\mathbf{R}^2\times\mathbf{R}^2, \frac{1}{100} < 2^j(1-|\xi'|) < 100 \qquad (2.18)\\
&\qquad \text{and } \frac{1}{100} < 2^j(1-|\xi''|) < 100\}, \\
|\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}\phi_{j,t}(\xi)| &\le C2^{|\alpha|j} \qquad \text{for all } |\alpha|\le 5. \qquad (2.19)
\end{aligned}
$$

Note that the bounds on the derivatives are uniform for $t\in[0,1]$, so both the above properties apply to $\int_0^1 \phi_{j,t}(\xi)dt$. The multiplier $-\lambda|\xi''|$, when smoothly cut off outside of the cylinder, represents a nice bounded multiplier operator so we can absorb it into a new term ϕ_j:

$$
\phi_j(\xi) = (-\lambda)|\xi''|\int_0^1 \phi_{j,t}(\xi)dt. \qquad (2.20)
$$

Since $|j-k|\le 1$, it will suffice to consider $j=k$. The $|j-k|=1$ cases will follow from the same proof. We have now reduced the multiplier to the simplified form

$$
\varphi(2^M h(\xi))\mu_{jk}(\xi) = 2^{-M}2^{-j(\lambda-1)}\widetilde{\varphi}\left(2^M h(\xi)\right)\varphi_j(\xi')\varphi_j(\xi'')\phi_j(\xi). \qquad (2.21)
$$

The $\varphi_j(\xi')\varphi_j(\xi'')\phi_j(\xi)$ portion is supported in a "annulus" a distance 2^{-j} from both the top and sides of the cylinder. It can be thought of as taking a two-dimensional square with sidelength 2^{-j}, rotating it around the x-axis through a third dimension, then rotating the result around the y-axis through a fourth dimension. The result is essentially a neighborhood of a torus imbedded in four dimensions. The $\varphi(2^M h(\cdot))$ term is supported a distance 2^{-M} from $|\xi'|=|\xi''|$. Together these are supported in a truncated "cone" of thickness 2^{-M} and "height" 2^{-j} in ξ' and ξ'' (see figure 1).

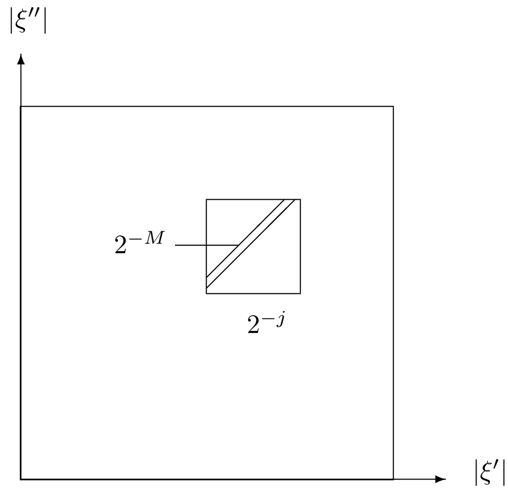

FIGURE 1. Support of the decomposition pieces

In a final decomposition, we introduce an equispaced cutoff χ. This will reduce the support from a cone to a sum of 2^{-M} neighborhoods of tori, with which analysis is much easier. Let $\chi \in \mathcal{C}_0^\infty$, $supp\chi \subset (-1,1)$, and $\sum_{N=-\infty}^{\infty} \chi(\cdot - N) \equiv 1$.

$$\begin{aligned}
\Phi_{Mj\nu}(\xi) &= \chi\left(2^{M(1+\epsilon)}\left(|\xi'| - \frac{\nu}{2^{M(1+\epsilon)}}\right)\right)\widetilde{\varphi}\left(2^M h(\xi)\right)\varphi_j(\xi')\varphi_j(\xi'')\phi_j(\xi),\\
\varphi(2^M h(\xi))\mu_{jk}(\xi) &= 2^{-M}2^{-j(\lambda-1)}\sum_{\nu\in\mathcal{V}}\Phi_{Mj\nu}(\xi).
\end{aligned}$$

The index set $\mathcal{V}$ represents the integers where $\Phi_{Mj\nu}$ is not identically zero. Due to supports of the various terms, the size of the set $\mathcal{V}$ is of the order $2^{M-j}2^{M\epsilon}$.

Lemma 2.3.

$$\|\Phi_{Mj\nu}\|_{M^4} \lesssim 2^{M\epsilon}. \tag{2.22}$$

Proof. The multiplier $\Phi_{Mj\nu}$ is a bump function supported on an $\mathbf{R}^2$ annulus crossed with another $\mathbf{R}^2$ annulus. We use a Taylor expansion to separate the ξ' and ξ'' variables, allowing us to apply the $\mathbf{R}^2$ Bochner-Riesz result twice.

We proceed by taking $\widetilde{\varphi}\left(2^M h(\xi)\right)\phi_j(\xi)$ and expanding it in a Taylor series in the $|\xi'|$ variable around $\frac{\nu}{2^{M(1+\epsilon)}}$. Denote $\chi_\nu(|\xi'|) = \chi\left(2^{M(1+\epsilon)}\left(|\xi| - \frac{\nu}{2^{M(1+\epsilon)}}\right)\right)$.

$$\begin{aligned}
\Phi_{Mj\nu}(\xi) &= \chi_\nu(|\xi'|)\varphi_j(\xi')\varphi_j(\xi'')\\
&\quad\cdot\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial|\xi'|^n}\left[\widetilde{\varphi}(2^M h(\xi))\phi_j(\xi)\right]\Big|_{|\xi'|=\frac{\nu}{2^{M(1+\epsilon)}}}\left(|\xi'| - \frac{\nu}{2^{M(1+\epsilon)}}\right)^n\\
&= \sum_{n=0}^{\infty}\chi_\nu(|\xi'|)\left(\varphi_j(\xi')\left(|\xi'| - \frac{\nu}{2^{M(1+\epsilon)}}\right)^n 2^{M(1+\epsilon)n}\right)\\
&\quad\cdot\left(\varphi_j(\xi'')\frac{1}{n!}\frac{\partial^n}{\partial|\xi'|^n}\left[\widetilde{\varphi}(2^M h(\xi))\phi_j(\xi)\right]\Big|_{|\xi'|=\frac{\nu}{2^{M(1+\epsilon)}}}2^{-Mn}\right)2^{-\epsilon n}.
\end{aligned}\tag{2.23}$$

Each derivative in the series loses 2^M but we gain $2^{-M(1+\epsilon)}$ due to the support of $\chi_\nu(|\xi'|)$. Therefore we need only consider the first N terms. The ξ' and ξ'' variables are now separate, and Theorem 2.1 applies to both the multipliers in the parentheses in line 2.23. The lemma follows immediately. $\square$

One more lemma is necessary to obtain some gain using the orthogonality of the χ_ν decomposition.

Lemma 2.4. *Let $\mathcal{V}$ be a finite interval in the integers $\mathbf{Z}$. Then for $2 \leq p \leq \infty$, $\frac{1}{p'} + \frac{1}{p} = 1$ and $f \in L^p$,*

$$\| \sum_{\nu \in \mathcal{V}} \mathcal{F}^{-1}[\chi_\nu] * f \|_{L^p} \lesssim \left(\sum_{\nu \in \mathcal{V}} \| \mathcal{F}^{-1}[\chi_\nu] * f \|_{L^p}^{p'} \right)^{\frac{1}{p'}} . \tag{2.24}$$

Proof. This lemma is easily verified for $p = 2$ using Plancherel's theorem.

The lemma follows by interpolation and duality from the $p = 2$ bound and the following $p = 1$ bound (triangle inequality):

$$\begin{aligned} \| \sum_{\nu} \mathcal{F}^{-1}[\chi_\nu] * f \|_{L^1} &\leq \sum_{\nu} \| \mathcal{F}^{-1}[\chi_\nu] * f \|_{L^1} \\ &= \left(\sum_{\nu} \| \mathcal{F}^{-1}[\chi_\nu] * f \|_{L^1}^{1} \right)^{1} . \end{aligned}$$

$\square$

Putting these to lemmas together allow us to finish Case (ii).

Lemma 2.5. *Let $\lambda > 0$. The following sums converge:*

$$\textstyle\sum_{j=2}^{\infty} \| m_{jj} \|_{M^4}, \tag{2.25}$$

$$\textstyle\sum_{j=2}^{\infty} \| m_{j,(j+1)} \|_{M^4}, \tag{2.26}$$

$$\textstyle\sum_{j=2}^{\infty} \| m_{(j+1),j} \|_{M^4}. \tag{2.27}$$

These sums represent all multiplier pieces where $|k - j| \leq 1$ and $k, j \neq 1$.

Proof. As before, we assume $j = k$. The other two cases follow similarly.

$$\|\mathcal{F}^{-1}[\varphi(2^M h(\xi))\mu_{jk}(\xi)] * f\|_{L^4} = 2^{-M}2^{-j(\lambda-1)}\|\sum_{\nu\in\mathcal{V}}\mathcal{F}^{-1}[\Phi_{Mj\nu}] * f\|_{L^4}$$

$$\leq 2^{-M}2^{-j(\lambda-1)}\left(\sum_{\nu\in\mathcal{V}}\|\mathcal{F}^{-1}[\Phi_{Mj\nu}] * f\|_{L^4}^{\frac{4}{3}}\right)^{\frac{3}{4}} \tag{2.28}$$

$$\lesssim 2^{-M}2^{-j(\lambda-1)}\left(\sum_{\nu\in\mathcal{V}}(2^{M\epsilon}\|f\|_{L^4})^{\frac{4}{3}}\right)^{\frac{3}{4}} \tag{2.29}$$

$$\lesssim 2^{-M}2^{-j(\lambda-1)}2^{M\epsilon}\left(2^{M-j}2^{M\epsilon}\|f\|_{L^4}^{\frac{4}{3}}\right)^{\frac{3}{4}} \tag{2.30}$$

$$= 2^{-\frac{M-j}{4}}2^{-j\lambda}2^{\frac{7}{4}M\epsilon}\|f\|_{L^4}, \tag{2.31}$$

$$\|\mathcal{F}^{-1}[\mu_{jj}] * f\|_{L^4} \lesssim \sum_{M=j-6}^{\infty}\|\mathcal{F}^{-1}[\varphi(2^M h(\xi))\mu_{jk}(\xi)] * f\|_{L^4} \tag{2.32}$$

$$\lesssim \sum_{M=j-6}^{\infty}2^{-\frac{M-j}{4}}2^{-j\lambda}2^{\frac{7}{4}M\epsilon}\|f\|_{L^4} \tag{2.33}$$

$$\lesssim 2^{-j\lambda}2^{\frac{7}{4}j\epsilon}. \tag{2.34}$$

The final term can be summed in j for $\lambda > \frac{7}{4}\epsilon$. Note the sum in M starts at $j-6$ due to the supports of φ_j and $\varphi(2^M h(\cdot))$. □

2.3. **Case (iii)** $|k-j| \leq 1$, $k = 1$ **or** $j = 1$.

Lemma 2.6.

$$\|\mathcal{F}^{-1}[m_{11} + m_{12} + m_{21}] * f\|_{L^p} \lesssim \|f\|_{L^p} \tag{2.35}$$

for all $f \in L^p(\mathbf{R}^d \times \mathbf{R})$, $1 < p < \infty$ *and* $\lambda > 0$.

Proof. In this region we use the following theorem of Seeger [6] to extend our result to the origin. The theorem applies with φ defined as in the rest of this paper.

Theorem 2.7 (Seeger)**.** *Suppose that* m *is a bounded function which satisfies for some* p, $1 < p < \infty$, $\epsilon > 0$

$$\sup_{t>0}\|\varphi(|\cdot|)m(t\cdot)\|_{M^p} \leq A \tag{2.36}$$

$$\sup_{t>0}\int_{|x|\geq w}|\mathcal{F}^{-1}[\varphi(|\cdot|)m(t\cdot)](x)|dx \leq B(1+w)^{-\epsilon} \tag{2.37}$$

Then

$$\|m\|_{M^p} \lesssim A[\log(2 + \frac{B}{A})]^{\frac{1}{p}-\frac{1}{2}}.$$

We begin by decompose the multiplier in this region similar to the way we did in (2.2). Note we work with μ_{11} and the other two multipliers are similar.

$$\mu_{11}(\xi) = \sum_{M=0}^{\infty}\varphi(2^M h(\xi))[(1-|\xi'|)^\lambda - (1-|\xi"|)^\lambda]\varphi_1(|\xi'|)\varphi_1(|\xi"|)$$

Denote the terms in the sum $\mu_M(\xi)$. By analyzing the multiplier $\varphi(|\cdot|)\mu_M(t\cdot)$, we can obtain boundedness results for μ_{11}. First we note that due to the support of μ_M, $\varphi(|\cdot|)\mu_M(t\cdot)$ is only non-zero if $t<2$. By analyzing derivatives of the multiplier and using the fact that t is bounded, we see that

$$\sup_{t>0}\|\varphi(|\cdot|)\mu_M(t\cdot)\|_{L^2_\alpha}\le C_\alpha 2^{-M}\qquad\text{independent of } t$$

We use this bound for the condition (2.36) in Theorem 2.7.

We obtain a bound of the form on (2.37) by doing integration by parts 6 times. Note that in the integration by parts we lose at most 2^M on each derivative since that is the loss in the multiplier's worst direction. We begin by defining the set Ω_j^ω.

$$\Omega_j^\omega=\{\xi=\{\xi_1,\xi_2,\xi_3,\xi_4\}\in\mathbf{R}^2\times\mathbf{R}^2,|\xi|>\omega\text{ and }|\xi_j|=\max_{k=1,\dots,4}|\xi_k|\}\tag{2.38}$$

$$\sup_{t>0}\int_{|x|>\omega}|\mathcal{F}^{-1}[\varphi(|\cdot|)\mu_M(t\cdot)](x)|dx\lesssim\sup_{t>0}\int_{|x|>\omega}\left|\int e^{ix\cdot\xi}\varphi(|\xi|)\mu_M(t\xi)d\xi\right|dx$$

$$\lesssim\ \sup_{t>0}\sum_{j=1}^4\int_{\Omega_j^\omega}\left|\int\frac{e^{ix\cdot\xi}}{|x_j|^6}\frac{\partial^6}{\partial\xi_j^6}(\varphi(|\cdot|)\mu_M(t\cdot))(\xi)d\xi\right|dx\tag{2.39}$$

$$\lesssim\ \sup_{t>0}\sum_{j=1}^4\int_{\Omega_j^\omega}\int\frac{2^{6M}}{|x_j|^6}\chi_{supp(\varphi(|\cdot|)\mu_M(t\cdot))}(\xi)d\xi dx\tag{2.40}$$

$$\lesssim\ \sup_{t>0}\sum_{j=1}^4\int_{\Omega_j^\omega}\frac{2^{5M}}{|x_j|^6}dx\tag{2.41}$$

$$\lesssim\ 2^{5M}(1+\omega)^{-\epsilon}\qquad\text{independent of } t\tag{2.42}$$

We then obtain a good bound on each μ_M and can sum in M.

$$\begin{aligned}\|\mu_M\|_{M^p}&\lesssim 2^{-M}\left[\log\left(2+\frac{2^{5M}}{2^{-M}}\right)\right]^{|\frac{1}{p}-\frac{1}{2}|}\\&\lesssim M^{|\frac{1}{p}-\frac{1}{2}|}2^{-M}\\\Rightarrow\|\mu_{11}\|_{M^p}&\lesssim\sum_{M=0}^{\infty}M^{|\frac{1}{p}-\frac{1}{2}|}2^{-M}\\&<\infty\end{aligned}$$

This bound on $\|\mu_{11}\|_{M^p}$ applies for $1<p<\infty$ and for all $\lambda>0$. □

Combining the results of the three cases completes the proofs for Theorem 1.1.

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