1. Find a point on the surface \( x^2 + 3y^2 + 2z^2 = 31 \) where the vector \( \mathbf{n} = \langle 1, 2, 4 \rangle \) is normal to the surface.

   **Solution.** For \( f(x, y, z) = x^2 + 3y^2 + 2z^2 \), the gradient \( \nabla f = \langle 2x, 6y, 4z \rangle \) is normal to the level surface \( f(x, y, z) = 31 \) at every point on the surface. Thus, we want \( \nabla f \) to be in the same direction of \( \mathbf{n} = \langle 1, 2, 4 \rangle \) at a point. Condition (a) means that we want \( \langle 2x, 6y, 4z \rangle = k\langle 1, 2, 4 \rangle \) and condition (b) means we want \( x^2 + 3y^2 + 2z^2 = 31 \). Combining these gives us \( k(2/3)^2 + 3(k/3)^2 + 2(k)^2 = 31 \), which is easy to solve as \( k = \pm \sqrt{12} \). Hence, the point \( (\sqrt{12}/2, \sqrt{12}/3, \sqrt{12}) \) will be on the surface and have the gradient in the direction of \( \mathbf{n} \).

2. Find a formula for the unit normal vector \( \mathbf{N}(t) \) for an object moving in a circle with radius 50m with constant speed 3 m/sec.

   **Solution.** The parametric curve \( \mathbf{r}(t) = \langle 50 \cos (3t/50), 50 \sin (3t/50) \rangle \) describes this kind of motion. In this case, \( \mathbf{r}'(t) = \langle -3 \sin (3t/50), 3 \cos (3t/50) \rangle \), which means that \( \| \mathbf{r}'(t) \| = 3 \), and so

   \[
   \mathbf{T}(t) = \langle -\sin (3t/50), \cos (3t/50) \rangle
   \]

   From here we find \( \mathbf{T}'(t) = \langle -3/50 \cos (3t/50), -3/50 \sin (3t/50) \rangle \), which means that \( \| \mathbf{T}'(t) \| = 3/50 \), and so

   \[
   \mathbf{N}(t) = \langle -\cos (3t/50), -\sin (3t/50) \rangle
   \]

3. A bug is located at the point \((-2, 4)\) on a hot plate that has temperature \( T(x, y) = 100e^{-(x^2+y^2)/1000} \) degrees \( C \) at any point \((x, y)\) measured in inches. The bug begins walking in a straight line toward the point \((2, 1)\) at a constant rate of 1 inch per second. At what rate is the bug’s temperature changing (in degrees per second) when the bug is at the point \((2, 1)\)?

   **Solution.** The motion of the bug can be parameterized as \( \mathbf{r}(t) = \langle -2 + 4t, 4 - 3t \rangle \) for \( 0 \leq t \leq 1 \), but this would have the bug traveling 5 inches in one second, so we use a unit vector for our direction vector instead and use the parameterization \( \mathbf{c}(t) = \langle -2 + \frac{4}{5}t, 4 - \frac{3}{5}t \rangle \) for \( 0 \leq t \leq 5 \). So

   \[
   \frac{\mathbf{d}}{dt} \left( \mathbf{T}(\mathbf{c}(t)) \right) = \nabla T(\mathbf{c}(t)) \cdot \mathbf{c}'(t)
   \]

   \[
   = \langle -\frac{x}{5} e^{-(x^2+y^2)/1000}, -\frac{y}{5} e^{-(x^2+y^2)/1000} \rangle \cdot \langle \frac{4}{5}, -\frac{3}{5} \rangle
   \]

   At \( t = 5 \) the bug is at the point \((2, 1)\) and this becomes

   \[
   \langle -\frac{2}{5} e^{-5/1000}, -\frac{1}{5} e^{-5/1000} \rangle \cdot \langle \frac{4}{5}, -\frac{3}{5} \rangle = -\frac{1}{5} e^{-5/1000} \approx -0.199 \text{ degrees per second.}
   \]

4. Use a linear approximation of the function \( f(x, y) = \frac{x^3}{\sqrt{y}} \) at the point \((2, 4)\) to give a paper-and-pencil estimate of the value of \( \frac{(2.05)^3}{\sqrt{4.1}} \).

   **Solution.** The linear approximation at the point \((2, 4)\) is given by

   \[
   L(x, y) = f(2, 4) + f_x(2, 4)(x-2) + f_y(2, 4)(y-4)
   \]

   Since \( f_x(x, y) = 3x^2y^{-1/2} \) and \( f_y(x, y) = -1/2x^3y^{-3/2} \), we have \( f_x(2, 4) = 6 \) and \( f_y(2, 4) = -\frac{1}{2} \), so the linear approximation is

   \[
   L(x, y) = 4 + 6(x-2) - \frac{1}{2}(y-4)
   \]

   Hence, \( f(2.05, 4.1) \approx L(2.05, 4.1) = 4 + 6(0.05) - \frac{1}{2}(0.1) = 4.25 \).
5. Demonstrate that the function \( u(x, y) = e^x \sin y \) satisfies Laplace equation \( u_{xx} + u_{yy} = 0 \).

Solution. Let \( u(x, y) = e^x \sin y \). Then \( u_x(x, y) = e^x \sin y \), and so \( u_{xx}(x, y) = e^x \sin y \). Also \( u(x, y) = e^x \cos y \), and so \( u_{yy}(x, y) = -e^x \sin y \). Therefore,

\[
 u_{xx} + u_{yy} = e^x \sin y + (-e^x \sin y) = 0
\]