Calculus III: Section 13.2

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Ship Math

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FIGURE 1 The vector-valued function $\mathbf{r}(t)$ approaches $\mathbf{u}$ as $t \to t_0$. 
**DEFINITION** Limit of a Vector-Valued Function  
A vector-valued function \( r(t) \) approaches the limit \( u \) (a vector) as \( t \) approaches \( t_0 \) if \( \lim_{t \to t_0} ||r(t) - u|| = 0 \). In this case, we write

\[
\lim_{t \to t_0} r(t) = u
\]

\[
r'(t) = \frac{d}{dt} r(t) = \lim_{h \to 0} \frac{r(t + h) - r(t)}{h}
\]
THEOREM 1 Vector-Valued Limits Are Computed Componentwise  A vector-valued function \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) approaches a limit as \( t \to t_0 \) if and only if each component approaches a limit, and in this case,

\[
\lim_{t \to t_0} \mathbf{r}(t) = \left( \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right)
\]
**Differentiation Rules**  
Assume that $\mathbf{r}(t)$, $\mathbf{r}_1(t)$, and $\mathbf{r}_2(t)$ are differentiable. Then

- **Sum Rule:** $(\mathbf{r}_1(t) + \mathbf{r}_2(t))' = \mathbf{r}_1'(t) + \mathbf{r}_2'(t)$
- **Constant Multiple Rule:** For any constant $c$, $(c \mathbf{r}(t))' = c \mathbf{r}'(t)$.
- **Product Rule:** For any differentiable scalar-valued function $f(t)$,

$$
\frac{d}{dt}(f(t)\mathbf{r}(t)) = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)
$$

- **Chain Rule:** For any differentiable scalar-valued function $g(t)$,

$$
\frac{d}{dt} \mathbf{r}(g(t)) = g'(t)\mathbf{r}'(g(t))
$$
Limits and Derivative

Example. Evaluate \( \lim_{t \to 0} \langle \cos t, e^{-t}, \frac{\sin t}{t} \rangle \).

Example. For \( \mathbf{r}(t) = \langle t^2, e^{-t}, \sin t \rangle \), evaluate \( \mathbf{r}'(t) \) and \( \frac{d}{dt} \left( e^t \mathbf{r}(t) \right) \).
THEOREM 3  Product Rule for Dot and Cross Products  
Assume that $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ are differentiable. Then

Dot Products:
$$\frac{d}{dt} \left( \mathbf{r}_1(t) \cdot \mathbf{r}_2(t) \right) = \mathbf{r}_1(t) \cdot \mathbf{r}'_2(t) + \mathbf{r}'_1(t) \cdot \mathbf{r}_2(t)$$

Cross Products:
$$\frac{d}{dt} \left( \mathbf{r}_1(t) \times \mathbf{r}_2(t) \right) = \left[ \mathbf{r}_1(t) \times \mathbf{r}'_2(t) \right] + \left[ \mathbf{r}'_1(t) \times \mathbf{r}_2(t) \right]$$

Example. For $\mathbf{r}_1(t) = \langle t^2, e^{-t}, \sin t \rangle$ and $\mathbf{r}_2(t) = \langle 3t, e^t, \cos t \rangle$, verify that Theorem 3 is true.
**THEOREM 3** **Product Rule for Dot and Cross Products**  
Assume that $r_1(t)$ and $r_2(t)$ are differentiable. Then

**Dot Products:** \[
\frac{d}{dt} (r_1(t) \cdot r_2(t)) = r_1(t) \cdot r_2'(t) + r_1'(t) \cdot r_2(t)
\]

**Cross Products:** \[
\frac{d}{dt} (r_1(t) \times r_2(t)) = [r_1(t) \times r_2'(t)] + [r_1'(t) \times r_2(t)]
\]

**Claim.** For any differentiable vector-valued function $r(t)$, show that

\[
\frac{d}{dt} (r(t) \times r'(t)) = r(t) \times r''(t)
\]
Product Rule

**THEOREM 3  Product Rule for Dot and Cross Products**  Assume that \( \mathbf{r}_1(t) \) and \( \mathbf{r}_2(t) \) are differentiable. Then

\[
\frac{d}{dt} (\mathbf{r}_1(t) \cdot \mathbf{r}_2(t)) = \mathbf{r}_1(t) \cdot \mathbf{r}_2'(t) + \mathbf{r}_1'(t) \cdot \mathbf{r}_2(t)
\]

\[
\frac{d}{dt} (\mathbf{r}_1(t) \times \mathbf{r}_2(t)) = [\mathbf{r}_1(t) \times \mathbf{r}_2'(t)] + [\mathbf{r}_1'(t) \times \mathbf{r}_2(t)]
\]

**Claim.** If the differentiable vector-valued function \( \mathbf{r}(t) \) has constant length, then \( \mathbf{r}(t) \) and \( \mathbf{r}'(t) \) are orthogonal.
Derivative is a tangent vector

\[ \mathbf{r}(t_0 + h) - \mathbf{r}(t_0) \]

\[ \frac{\mathbf{r}(t_0 + h) - \mathbf{r}(t_0)}{h} \]

**Figure 2** The difference quotient points in the direction of \( \Delta \mathbf{r} = \mathbf{r}(t_0 + h) - \mathbf{r}(t_0) \).
Derivative is a tangent vector

\[
\frac{\mathbf{r}(t_0 + h) - \mathbf{r}(t_0)}{h}
\]

\(h\) tending to zero

limit as \(h \to 0\)

\[
\mathbf{r}'(t_0)
\]

**FIGURE 3** The difference quotient converges to a vector \(\mathbf{r}'(t_0)\), tangent to the curve.
Derivative is a tangent vector

Example. Find the parametric equation of the tangent lines shown above.

http://www.flashandmath.com/advanced/motion3d/motion3d.html
Consider the cycloid \( r(t) = \langle t - \sin t, 1 - \cos t \rangle \). Find values of \( t \) for which \( r'(t) = 0 \).

**Figure 5** Points on the cycloid
Recall. If the differentiable vector-valued function $\mathbf{r}(t)$ has constant length, then $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal.
**THEOREM 4**       If $\mathbf{R}_1(t)$ and $\mathbf{R}_2(t)$ are differentiable and $\mathbf{R}_1'(t) = \mathbf{R}_2'(t)$, then

$$\mathbf{R}_1(t) = \mathbf{R}_2(t) + \mathbf{c}$$

for some constant vector $\mathbf{c}$.

**Fundamental Theorem of Calculus for Vector-Valued Functions**       If $\mathbf{r}(t)$ is continuous on $[a, b]$, and $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$, then

$$\int_a^b \mathbf{r}(t) \, dt = \mathbf{R}(b) - \mathbf{R}(a)$$
Example. If \( r'(t) = \langle 100, 200, 300 - 32t \rangle \) and \( r(0) = \langle 0, 0, -100 \rangle \), then find \( r(t) \).

Example. Evaluate

\[
\int_0^1 \left\langle \frac{t}{t^2 + 1}, \sin t \cos t \right\rangle \, dt
\]