Tangent Plane

**FIGURE 1** The tangent plane to a surface.
FIGURE 2 The tangent lines to the trace curves $C_1$ and $C_2$ lie in the tangent plane (and span it).
Example (14.4 #1) Find the equation of the plane tangent to $f(x, y) = x^2y^3$ at the point $(2, 1, 4)$. Use this to get an approximation for the value of $f(2.01, 1.02) = (2.01)^2(1.02)^3$. 

$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$
**DEFINITION** Local Linearity  
Assume that $f(x, y)$ is defined in a disk $D$ containing $(a, b)$ and that $f_x(a, b)$ and $f_y(a, b)$ exist. We say that $f(x, y)$ is **locally linear** at $(a, b)$ if $L(x, y)$ approximates $f(x, y)$ at $(a, b)$ to **first order**, that is, if

$$f(x, y) = L(x, y) + \varepsilon(x, y)\sqrt{(x - a)^2 + (y - b)^2}$$

for $(x, y) \in D$

where $\varepsilon(x, y)$ is a function such that $\lim_{(x,y)\to(a,b)} \varepsilon(x, y) = 0$.

**Example** Find the equation of the plane tangent to $f(x, y) = 5x + 4y^2$ at the point $P = (2, 1, 14)$. (Note that Example 2 in Section 14.4 proves that this answer satisfies the definition of “local linearity.”)
**Figure 3** Graph of $f(x, y) = 5x + 4y^2$ and the tangent plane at $P = (2, 1, 14)$. 
**DEFINITION**  Differentiability and the Tangent Plane  
Assume that \( f(x, y) \) is defined in a disk \( D \) containing \((a, b)\). We say that \( f(x, y) \) is **differentiable** at \((a, b)\) if:

- \( f_x(a, b) \) and \( f_y(a, b) \) exist, and
- \( f(x, y) \) is locally linear at \((a, b)\).

In this case, the **tangent plane** to the surface \( z = f(x, y) \) at \((a, b, f(a, b))\) is the plane with equation \( z = L(x, y) \). Explicitly,

\[
z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)
\]

**THEOREM 1**  Criterion for Differentiability  
If \( f_x(x, y) \) and \( f_y(x, y) \) exist and are continuous on an open disk \( D \), then \( f(x, y) \) is differentiable on \( D \).
**FIGURE 4** The function $f(x, y) = x \sin(x + y)$ is differentiable.
The function $h(x, y) = \sqrt{x^2 + y^2}$ is differentiable except at the origin.
**FIGURE 7** The graph of $g(x, y) = \frac{2xy(x + y)}{x^2 + y^2}$. The partial derivatives at $(0, 0)$ exist, but the graph is not locally linear at $(0, 0)$.
Linear Approximation

\[ \Delta f \approx f_x(a, b) \Delta x + f_y(a, b) \Delta y \]

**Example** (Section 14.4 #32) The body mass index (BMI) is defined by the function \( f(W, H^2) = \frac{W}{H^2} \), where \( W \) and \( H \) are weight and height, respectively. Estimate the change in BMI if a boy with weight 34 kg and height 1.3m grows to be 1.32m and 36kg.

**Example** (Section 14.4 #32) Use a linear approximation to estimate the value of \( \frac{8.01}{\sqrt{(1.99)(2.01)}} \).