Calculus III: Section 14.8

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Ship Math

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**FIGURE 1** Optimization with a constraint: find the minimum of \( f(x, y) = \sqrt{x^2 + y^2} \) on the line \( x + y = 4 \).
(A) The gradient vector $\nabla f_Q$ shows that $f$ increases as we move to the right along the constraint curve.

(B) The local maximum of $f$ on the constraint curve $g(x, y) = 0$ occurs at a point $P$ where $\nabla f_P$ and $\nabla g_P$ point in the same direction.

**FIGURE 2**
THEOREM 1 Lagrange Multipliers Assume that $f(x, y)$ and $g(x, y)$ are differentiable functions. If $f(x, y)$ has a local minimum or maximum on the constraint curve $g(x, y) = 0$ at $P = (a, b)$, and if $\nabla g_P \neq 0$, then there is a scalar $\lambda$ such that

$$\nabla f_P = \lambda \nabla g_P$$

Proof Let $c(t)$ be a parametrization of the constraint curve $g(x, y) = 0$ near $P$ such that $c(0) = P$ and $(a, b)$ and $c'(0) \neq 0$. Then $f(c(0)) = f(P)$, and $f(c(t))$ has a local minimum or maximum at $t = 0$. Thus, $t = 0$ is a critical point of $f(c(t))$ and

$$\left. \frac{d}{dt} f(c(t)) \right|_{t=0} = \nabla f_P \cdot c'(0) = 0$$

Chain Rule

This shows that $\nabla f_P$ is orthogonal to the tangent vector $c'(0)$ to the curve $g(x, y) = 0$. The gradient $\nabla g_P$ is also orthogonal to $c'(0)$ since $\nabla g_P$ is orthogonal to the level curve $g(x, y) = 0$ at $P$. We conclude that $\nabla f_P$ and $\nabla g_P$ are proportional as claimed.
**FIGURE 3** The minimum and maximum occur where the level curve of $f(x, y) = x + 2y$ is tangent to the constraint curve $3x^2 + 4y^2 = 3$. 
Examples

Figure 5 Contour plot of the Cobb–Douglas production function $P(x, y) = 50x^{0.4}y^{0.6}$. 
\textbf{FIGURE 6} Point $P$ closest to the origin on the plane.