Line Integrals

Partition of \( C \) into \( N \) small arcs

Choice of sample points \( P_i \) in each arc

**FIGURE 1** The curve \( C \) divided into \( N \) small arcs.

\[ P_i = \mathbf{c}(t_i^*) \]

**FIGURE 2** Partition of parametrized curve \( \mathbf{c}(t) \).
THEOREM 1  **Computing a Scalar Line Integral**  Let \( c(t) \) be a parametrization of a curve \( C \) for \( a \leq t \leq b \). Assume that \( f(x, y, z) \) and \( c'(t) \) are continuous. Then

\[
\int_C f(x, y, z) \, ds = \int_a^b f(c(t)) \|c'(t)\| \, dt
\]

The value of the integral on the right does not depend on the choice of parametrization. For \( f(x, y, z) = 1 \), we obtain the length of \( C \):

\[
\text{Length of } C = \int_C \|c'(t)\| \, dt
\]
Example 1-ish. Find $\int_C (xy + z^2) \, ds$, where $C$ is the helix $\mathbf{c}(t) = \langle \cos t, \sin t, t \rangle$ with $0 \leq t \leq 2\pi$. 

**FIGURE 3** The helix $\mathbf{c}(t) = (\cos t, \sin t, t)$. 
Example 2. Find the total mass of a wire in the shape of the parabola $y = x^2$ for $1 \leq x \leq 4$ if the mass density at any point $(x, y)$ is given by $\rho(x, y) = \frac{y}{x}$ grams per centimeter.
Line Integrals

Oriented path from $P$ to $Q$

**FIGURE 5** An oriented curve is a curve with a specified direction.

A closed oriented path
**DEFINITION Vector Line Integral**  Let $C$ be an oriented curve and let $T$ denote the unit tangent vector pointing in the forward direction along $C$. The line integral of a vector field $\mathbf{F}$ along $C$ is the integral of the tangential component of $\mathbf{F}$:

$$\int_C \mathbf{F} \cdot ds = \int_C (\mathbf{F} \cdot T) \, ds$$
**Figure 6** The line integral is the integral of the tangential component of $\mathbf{F}$ along $\mathcal{C}$. 

$\mathbf{F} \cdot \mathbf{T}$ is the length of the projection of $\mathbf{F}$ along $\mathbf{T}$. 

$\mathbf{c}(a)$

$\mathbf{c}(b)$
**Theorem 2** Computing a Vector Line Integral  
Let \( \mathbf{c}(t) \) be a regular parametrization of an oriented curve \( C \) for \( a \leq t \leq b \). The line integral of a vector field \( \mathbf{F} \) over a curve \( C \) is equal to

\[
\int_C \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) \, dt
\]

**Exercise #8.** Let \( \mathbf{F} = \langle xy, 2, z^3 \rangle \), and evaluate \( \int_C \mathbf{F} \cdot d\mathbf{s} \), where \( C \) is parameterized by \( \mathbf{c}(t) = \langle \cos t, \sin t, t \rangle \) for \( 0 \leq t \leq \pi \).
The dot product $\mathbf{T} \cdot \mathbf{F}$ is negative because the angle between the vectors is obtuse.

Here, the dot product $\mathbf{T} \cdot \mathbf{F}$ is positive because the angle between the vectors is acute.

(A) Line integral is negative

(B) Line integral is positive

(C) Total line integral is negative

**FIGURE 7** The vector field $\mathbf{F} = \langle 2y, -3 \rangle$. 

---

Professor Ensley  (Ship Math)
Exercise #13

(A)  

(B)  

(C)
Exercise #14

(A)

(B)

(C)
Figure 8: The path from $P$ to $Q$ has two possible orientations.
THEOREM 3 Properties of Line Integrals

Let $C$ be a smooth oriented curve and let $\mathbf{F}$ and $\mathbf{G}$ be vector fields.

(i) Linearity: $\int_C (\mathbf{F} + \mathbf{G}) \cdot ds = \int_C \mathbf{F} \cdot ds + \int_C \mathbf{G} \cdot ds$

$\int_C k\mathbf{F} \cdot ds = k \int_C \mathbf{F} \cdot ds$ (for $k$ a constant)

(ii) Reversing orientation: $\int_{-C} \mathbf{F} \cdot ds = -\int_C \mathbf{F} \cdot ds$

(iii) Additivity: If $C$ is a union of $n$ smooth curves $C_1 + \cdots + C_n$, then

$\int_C \mathbf{F} \cdot ds = \int_{C_1} \mathbf{F} \cdot ds + \cdots + \int_{C_n} \mathbf{F} \cdot ds$
Example

Exercise #41. Let \( \mathbf{F} = \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) \), the vortex vector field. Calculate \( \int_C \mathbf{F} \cdot d\mathbf{s} \), where \( C \) is a circle of radius 2 centered at the origin oriented counterclockwise. (Before you do the calculation, do you predict the answer will be positive, negative, or zero?)
Exercise #49. Calculate the work done by the force field \( \mathbf{F} = \langle x, y, z \rangle \) along the helix path \( \mathbf{c}(t) = \langle \cos t, \sin t, t \rangle \) for \( 0 \leq t \leq 3\pi \).