Calculus III: Section 17.1

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Ship Math

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**FIGURE 1** The boundary of the domain \( \mathcal{D} \) is a simple closed curve \( \mathcal{C} \), oriented in the counterclockwise direction.
Green’s Theorem

**Theorem 1** Green’s Theorem \[ \oint_{\partial D} P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \]

**Alternate Notation** Instead of \[ F \cdot c'(t), \] we can think of \[ F = \langle P, Q \rangle \text{ and } c'(t) = \langle \frac{dx}{dt}, \frac{dy}{dt} \rangle, \] and then we can write

\[
\int F \cdot c'(t) \, dt = \int P \, dx + Q \, dy
\]
Verify Green’s Theorem

**Example 1** Evaluate $\oint_C xy^2 \, dx + x \, dy$ where $C$ is the unit circle oriented counterclockwise.

**Figure 3** The vector field $\mathbf{F} = (xy^2, x)$. 
Exercise 9 Use Green’s Theorem to evaluate

\[ \int F \cdot ds \]

where \( F = \langle x^3, 4x \rangle \) and \( C \) is the clockwise path around the boundary of the parallelogram with vertices \((0, 0), (2, 2), (4, 2), \) and \((2, 0)\).
Use Green’s Theorem to Evaluate an Area

Area enclosed by $C = \frac{1}{2} \oint_C x \, dy - y \, dx$

**Example 3** Use the formula above to find the area enclosed by the ellipse
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad (\text{Use counterclockwise parameterization } \langle a \cos t, b \sin t \rangle \text{ for } 0 \leq t \leq 2\pi.)
\]
**FIGURE 7** The circulation of $\mathbf{F}$ around $\partial \mathcal{R}$ is approximately $\text{curl}_z(\mathbf{F})(P) \cdot \text{Area}(\mathcal{R})$. 
FIGURE 8 The curl at $P$ is approximately equal to one-half the angular velocity (in radians per unit time) of a small paddle wheel placed at $P$. 
Curl

(A) \( \mathbf{F} = \langle -y, x \rangle \)
\[ \text{curl}_z(\mathbf{F}) = 2 \]

(B) \( \mathbf{F} = \langle y - x, -x - y \rangle \)
\[ \text{curl}_z(\mathbf{F}) = -2 \]

(C) \( \mathbf{F} = \langle y, 0 \rangle \)
\[ \text{curl}_z(\mathbf{F}) = -1 \]

(D) \( \mathbf{F} = \langle y, x \rangle \)
\[ \text{curl}_z(\mathbf{F}) = 0 \]

(E) \( \mathbf{F} = \langle x, y \rangle \)
\[ \text{curl}_z(\mathbf{F}) = 0 \]

**FIGURE 9**
Additivity

\[ \oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \oint_{\partial D_1} \mathbf{F} \cdot d\mathbf{s} + \oint_{\partial D_2} \mathbf{F} \cdot d\mathbf{s} \]
Additivity

We can use this idea to show in the figure below that

$$\int_{C_1} \mathbf{F} \cdot ds = \int_{C_2} \mathbf{F} \cdot ds + \iint_D \text{curl}_z(\mathbf{F}) dA$$

**FIGURE 13** $\mathcal{D}$ has area 8 and $C_1$ is a circle of radius 1.