Axiomatic Systems

An axiomatic system is a list of undefined terms together with a list of statements (called "axioms") that are presupposed to be "true." A theorem is any statement that can be proven using logical deduction from the axioms.

Examples

Here are some examples of axiomatic systems.

- Committees

  Undefined terms: committee, member
  Axiom 1: Each committee is a set of three members.
  Axiom 2: Each member is on exactly two committees.
  Axiom 3: No two members may be together on more than one committee.
  Axiom 4: There is at least one committee.

- Monoid

  Undefined terms: element, product of two elements
  Axiom 1: Given two elements x and y, the product of x and y, denoted $x * y$, is a uniquely defined element.
  Axiom 2: Given elements x, y, and z, the equation $x * (y * z) = (x * y) * z$ is always true.
  Axiom 3: There is an element e, called the identity, such that $e * x = x$ and $x * e = x$ for all elements x.

- Silliness

  Undefined terms: silly, dilly.
  Axiom 1: Each silly is a set of exactly three dillies.
  Axiom 2: There are exactly four dillies.
  Axiom 3: Each dilly is contained in a silly.
  Axiom 4: No dilly is contained in more than one silly.

Notice that in the second example, the axioms defined a new term ("identity"). This isn't an undefined term because the axiom includes a definition. Also, these axioms refer to basic set theory that you learned in Discrete Math. For our purposes, we will assume all of those basic set theory terms are known. It is possible to view set theory itself as another axiomatic system, but that is beyond the scope of this course.

Models

A model for an axiomatic system is a way to define the undefined terms so that the axioms are true. Sometimes it is easy to find a model for an axiomatic system, and sometimes it is more difficult.

Here are some examples of models for the "monoid" system.

- the elements are real numbers, and the product of two elements is the product of those two numbers
- the elements are 2$\times$2 matrices, and the product is the product of those two matrices
- the elements are integers, and the "product" of two elements is actually the sum of the two elements (replace the * symbol with + and check that all of the axioms still work).

We have to be careful not to assume that the axioms say things that they do not. Even though the undefined term is called "product," there is nothing in the axioms that prevents us from using addition to stand for that term.

- **Discussion Question**: Can you think of a new axiom that would exclude addition from being the "product" of a monoid?

Here is a model for the Committees system (but certainly not the only one):

Members	Alan, Beth, Chris, Dave, Elena, Fred
Committees	{Alan, Beth, Chris} {Alan, Dave, Elena} {Beth, Dave, Fred} {Chris, Elena, Fred}

We have defined the undefined terms, and now we have to check that the axioms are actually satisfied. It is easy to see that Axioms 1 and 4 are satisfied.

Axiom 2 says "Each member is on exactly two committees." To check this axiom, we look at each member, and list the number of committees they are on. If that number is 2 for every member, then the axiom is true.

Member	*Committees*	*Number = 2?*
Alan	{**Alan**, Beth, Chris}, {**Alan**, Dave, Elena}	yes
Beth	{Alan, **Beth**, Chris}, {**Beth**, Dave, Fred}	yes
Chris	{Alan, Beth, **Chris**}, {**Chris**, Elena, Fred}	yes
Dave	{Alan, **Dave**, Elena}, {Beth, **Dave**, Fred}	yes
Elena	{Alan, Dave, **Elena**}, {Chris, **Elena**, Fred}	yes
Fred	{Beth, Dave, **Fred**}, {Chris, Elena, **Fred**}	yes

Axiom 3 says "No two members may be together on more than one committee." For this axiom, we have to look at all 15 pairs of members and make sure that none of the pairs is on more than one committee. So it is acceptable to have the pair of members be on zero committees or one committee, but not two or more.

Pair of Members	*Committee(s)*	*Number ≤ 1?*
Alan & Beth	{**Alan**, **Beth**, Chris}	yes
Alan & Chris	{**Alan**, Beth, **Chris**}	yes
Alan & Dave	{**Alan**, **Dave**, Elena}	yes
Alan & Elena	{**Alan**, Dave, **Elena**}	yes
Alan & Fred	*none*	yes
Beth & Chris	{Alan, **Beth**, **Chris**}	yes
Beth & Dave	{**Beth**, **Dave**, Fred}	yes
Beth & Elena	*none*	yes
Beth & Fred	{**Beth**, Dave, **Fred**}	yes
Chris & Dave	*none*	yes
Chris & Elena	{**Chris**, **Elena**, Fred}	yes
Chris & Fred	{**Chris**, Elena, **Fred**}	yes
Dave & Elena	{Alan, **Dave**, **Elena**}	yes
Dave & Fred	{Beth, **Dave**, **Fred**}	yes
Elena & Fred	{Chris, **Elena**, **Fred**}	yes

Independence

An axiom is called independent if it cannot be proven from the other axioms. In other words, the axiom "needs" to be there, since you can't get it as a theorem if you leave it out. How do you prove that something *can't* be proved? This relates to the area of mathematics known as *logic*.

Consider **Axiom 1** from the Committee system. Let's omit it and see what kind of model we can come up with.

Members	Adam, Brian, Carla, Dana
Committees	{Adam, Brian} {Brian, Carla, Dana} {Adam, Carla} {Dana}

Notice that we found a model where Axiom 1 is not true; we have committees that do not have exactly three members. Since all of the other axioms are true in this model, then so is any statement that we could prove using those axioms. But since Axiom 1 is not true, it follows that Axiom 1 is not provable from the other axioms.

To prove that one axiom is independent from all of the others, find a model in which the axiom is false, but all of the other axioms are true.

- **Discussion Question.** Is Axiom 2 of the Committee system independent from Axioms 1, 3, and 4? If it is, you should be able to come up with a model where Axiom 2 is false, but Axioms 1, 3, and 4 are all true.

Consistency

If there is a model for an axiomatic system, then the system is called consistent. Otherwise, the system is inconsistent. In order to prove that a system is consistent, all we need to do is come up with a model: a definition of the undefined terms where the axioms are all true. In order to prove that a system is inconsistent, we have to somehow *prove* that no such model exists (this is much harder!).

The Silliness axiomatic system is an example of an inconsistent system. Here is a proof of that fact. If you find the language confusing, try replacing the word "dilly" with "element" and the word "silly" with "set."

Proof: Assume that there is a model for the Silliness axiomatic system. By Axiom 2, there are four dillies. Let *a* be a dilly. By Axiom 3, *a* is contained in a silly, which is a set of dillies. By Axiom 1, this silly contains two other dillies, say *b* and *c*. By Axiom 2, there is only one other dilly; we'll call it *d*. Now Axiom 3 tells us that *d* is contained in a silly, but it's not contained in the silly {*a*, *b*, *c*}. So *d* must be contained in a new silly. By Axiom 1, this silly must contain three dillies. But Axiom 4 prevents this new silly from containing *a*, *b*, or *c*. Since these are the only dillies that there are, we have reached a contradiction. The new silly must contain three dillies, but there is only one remaining. ■

Completeness

An axiomatic system is complete if every true statement can be proven from the axioms. What does it mean for a statement to be true but not provable? Consider this example:

Twin Primes Conjecture: There are an infinite number of pairs of primes whose difference is 2.

Some examples of "twin" primes are 3 and 5, 5 and 7, 11 and 13, 101 and 103, etc. Computers have found very large pairs of twin primes, but so far no one has been able to prove this theorem. It is possible that a proof will never be found. In fact, in 2004, a proof was claimed to have been discovered, but a serious flaw in the proof was found and the problem remains unsolved.

The Hilbert Program

In 1900, a famous mathematician named David Hilbert set out a list of 23 unsolved mathematical problems to focus the direction of research in the 20th Century. Many of these problems remain unsolved to this day. Hilbert's Second Problem challenged mathematicians to prove that mathematics itself could be reduced to a consistent set of axioms that was complete. In other words, the problem was to find axioms from which all mathematical truths could be proven.

In 1930, a mathematician named Kurt Gödel proved the Incompleteness Theorem. Basically, the theorem says that in any "sufficiently complex" consistent axiomatic system, there must exist true statements that cannot be proven. Here "sufficiently complex" basically means anything robust enough to be able to describe arithmetic (including addition and multiplication, prime numbers, divisibility, etc.).

So Hilbert's Second Problem was solved, but certainly not in the way he intended. By Gödel's theorem, we now know that mathematics necessarily contains true statements for which a proof can never be found.

Exercises

Consider the following axiomatic system for Bus Routes.

Undefined Terms: route, stop
Axiom 1: Each route is a list of stops in a particular order. These stops are called the stops *visited by* the route.
Axiom 2: Each route visits at least four distinct stops.
Axiom 3: No route visits the same stop twice, except for the first stop, which is always the same as the last stop.
Axiom 4: There is a stop called *downtown* that is visited by each route.
Axiom 5: Every stop other than downtown is visited by at most two routes.

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1. Construct a model of this system with three routes. What is the fewest number of stops you can use?

2. Your answer for Problem 1 shows that this system is … (choose one)

   (a) complete

   (b) consistent

   (c) inconsistent

   (d) independent

3. Consider the following model. Is it a model for the Bus Routes system? If not, determine which axioms are satisfied by the model and which are not.

Stops	Downtown, Queen St., Wal-Mart, Sheetz, Giant, CVS, King St.
Routes	**Route 1:** Downtown, Wal-Mart, King St., Queen St., Downtown **Route 2:** King St., Queen St., Sheetz, Giant, Downtown, King St. **Route 3:** Wal-Mart, King St., Downtown, Giant, King St., Wal-Mart

4. Show that **Axiom 3** is independent from the other axioms.

5. Demonstrate that “There are exactly three routes” is *not* a theorem in this system by finding a model in which it is not true.

Possible Solutions

1. Answers may vary. You need at least 6 stops.

2. (b) consistent.

3. It is not a model for this system.

   - Axioms 1, 2, and 4 are satisfied.

   - Axiom 3 is not satisfied because Route 3 visits King St. twice and it is not the first/last stop.

   - Axiom 5 is not satisfied because all three routes visit King St.

4. To do this, we must construct a model in which Axiom 3 is false but the other axioms are true. Answers may vary. (Note that the model given in Problem 3 does not suffice, since *both* Axioms 3 and 5 are false in that model.)

5. Answers may vary. Construct a model containing either more or fewer than three routes.