D. Mirimanoff, A propos de l'interprétation géométrique du problème du scrutin, *L'enseignement mathématique* 23 (1923) 187–189.

On the geometric interpretation of the ballot problem

by

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It seems to me that it would not be useless to show how the solution given by Mr. Aebly in the note above differs from the solution of D. André which Bertrand, Poincaré and Mr. Czuber reproduced in their treaties.

It is known that the reasoning of André rests on the following lemma: the number of unfavorable sequences starting with A is equal to that of the sequences starting with B. Mr. Aebly succeeded in simplifying the proof of this lemma by introducing a particular correspondence, which was suggested to him by his geometrical interpretation of the problem. Instead of splitting off the segment for which the equality of the votes occurs for the first time and transporting the removed part to the other end of the sequence, Mr. Aebly replaces the segment by its reflection which one obtains by applying the transposition (A, B) to the letters of the segment.

Let us take, for example, the sequence

$$AABABBABAA$$

under consideration by Poincaré.

The segment for which the equality of the votes occurs for the first time is formed of the first six letters $AABABB$.

In André's solution, this segment is split into two: one leaves the last letter in its place and transports the first five to the right side [of the entire sequence], which creates the sequence

$$BABAAAABAB.$$

Mr. Aebly instead applies the transposition (A, B) to the letters of the segment; the associated sequence is written

$$BBABAAABAA.$$

Conversely one passes from a sequence starting with B to an unfavorable sequence starting with A by applying the same transposition (A, B) to the letters of the first segment containing equal votes. It is seen immediately that this correspondence is bijective.

I pass to the geometric interpretation of the ballot problem which consists in representing the various sequences by paths traversed on a rectangular chess-board. There would perhaps be some interest in relating this topological problem to another rather curious problem which was posed recently by a maker of fountain pens. [Translator's note: we are unsure of this last sentence.]

But I would like better, before finishing, to show how the consideration of these paths can simplify the demonstration of some fundamental properties of the binomial coefficients.

Let us start, as Mr. Aebly does, from the box $(0, 0)$; let us indicate by N_{ik} the number of the paths which end in the box (i, k). It is seen immediately that $N_{ik} = N_{i-1,k} + N_{i,k-1}$, since any path leading to (i, k) necessarily passes through the box $(i-1, k)$ or through the box $(i, k-1)$. It results that N_{ik} are the numbers in Pascal's triangle, i.e., the binomial coefficients. Additionally, the formula for the permutations with repetition gives $\frac{(i+k)!}{i!k!}$. One draws from this the traditional expression for the binomial coefficients.

But one can go further in this direction. Consider the paths which end in a given box (n, m). Each one of these paths crosses the diagonal along which the sum $i + k$ of the indices equals s, where s is a given number less than $m + n$.

Consequently, $N_{n,m}$ is equal to the sum of the numbers of the paths passing through the various boxes of this diagonal. However, the number of the paths passing through a box (i, k) and leading to the final box (n, m) is equal to the product of $N_{i,k}$ by the number of the paths from (i, k) to (n, m), i.e. by $N_{n-i,m-k}$. It results that the sum of these products is equal to $N_{n,m}$. In particular, one obtains in this manner the expression for the sum of the squares of the binomial coefficients. These formulas are known, and one finds them, for example, in a book of P. Bachmann[1]. Is the demonstration that I have just given new? I do not believe it, but I thought that it was not useless to indicate it.

[1]P. Bachmann, Niedere Zahlentheorie, II, Teubner, 1910, p. 122.