Stupid Divisibility Tricks

101 Ways to Stupefy Your Friends

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1 Introduction

When is the last time this happened to you? You are stranded on a deserted island without a calculator and for some reason you must determine if 67 is a divisor of 95733553; furthermore, a coconut recently fell on your head and you have completely forgotten how to perform long division.

Of course the above scenario would never happen (we all carry around calculators) but it's good to know that if we should find ourselves in a similar situation there is an easy divisibility rule for 67: remove the two rightmost digits from the number (in our case, 53), double them (106) and subtract that from the remaining digits ($957335 - 106 = 957229$); the original number is divisible by 67 if and only if the resulting number is divisible by 67. If the resulting number is not obviously divisible by 67 we can repeat the process until we get a number that clearly is or is not a multiple of 67. In the above example, we get the following.

$$
\begin{array}{rr}
 & 957335\not{5}\not{3} \\
- & 106 \\
\hline
 & 9572\not{2}\not{9} \\
- & 58 \\
\hline
 & 95\not{1}\not{4} \\
- & 28 \\
\hline
 & 67
\end{array}
$$

Thus we conclude that in fact 95733553 is a multiple of 67.

This article has two aims. First, to identify six categories of tests that most divisibility tricks fall into, and second, to provide an easy divisibility test for each number from 2 – 102 (thus the "101 Ways..." in the title). We'll see that in fact many numbers have more than one divisibility test.

Divisibility tests have always fascinated people. Many of us learn "the rule of three" in childhood: a number is divisible by 3 if and only if the sum of its digits is divisible by 3. The Babylonians knew that a number of the form $100a + b$ is divisible by 7 if and only if $2a + b$ is divisible by 7. Chapter 12 of L. E. Dickson's classic 1919 text *History of the Theory of Numbers* is entitled "Criteria for Divisibility by a Given Number" and contains a collection of divisibility tests gathered throughout history and covering many cultures. In a 1962 *Scientific American* article Martin Gardner discusses divisibility rules for 2 – 12, and he explains that the rules were widely known during the Renaissance and used to reduce fractions with large numbers down to lowest terms. Today, most modern number theory textbooks present a few divisibility tests and explain why they work; a quick search on the Internet uncovers many articles that treat divisibility by the numbers 2 – 12, and a few that address divisibility by the primes 13, 17, and 19.

Disclaimer: Let's be honest – these tests aren't particularly practical in this age of the graphing calculator and laptop computer. Moreover, long division is often just as fast *and* you end up knowing the quotient and remainder as well. However, there is something intriguing about the fact that you can test divisibility by 3 by adding all the digits or you can test divisibility by 67 as outlined above, and it is this aspect of divisibility that motivates this article.

2 Modular Arithmetic

Modular arithmetic is the tool that allows us to find and analyze divisibility tests. Let a and b be integers, and let m be a positive integer. We say that a and b are *congruent modulo* m (or a is congruent to b modulo m) if a and b both leave the same remainder when divided by m, and we write this mathematically as $a \equiv b \pmod{m}$. For example,

$$13 \equiv 22 \pmod{3} \qquad 8 \equiv 0 \pmod{4} \qquad 14 \equiv -1 \pmod{5}.$$

Equivalently, $a \equiv b \pmod{m}$ if $a-b$ is a multiple of m. When $n \equiv 0 \pmod{d}$ we say that n is divisible by d, or d divides n. There are two facts about modular arithmetic that will be particularly helpful.

1. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a + c \equiv b + d \pmod{m}$.
2. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$.

For example, $36 \equiv 1 \pmod{5}$ and $9872 \equiv 2 \pmod{5}$, so by the first property, $36 + 9872 \equiv 1 + 2 = 3 \pmod{5}$, and by the second property, $(36)(9872) \equiv (1)(2) = 2 \pmod{5}$. What is the remainder when 324^{3847} is divided by 5?

Since $324 \equiv -1 \pmod 5$ we have

$$324^{3847} \equiv (-1)^{3847} = -1 \equiv 4 \pmod 5$$

and so the remainder is 4.

3 The Divisibility Tests

In our base 10 number system, the number a composed of the digits a_k, a_{k-1}, $\ldots$, a_1, a_0 read from left to right can be written as the sum

$$a = 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10a_1 + a_0. \tag{1}$$

Our standard method for testing the divisibility of a by d is to reduce the above sum modulo d and see what information we get.

For ease of notation, we will write $[a_k a_{k-1} \ldots a_1 a_0]$ to denote the number whose (base 10) digits are $a_k,\ a_{k-1},\ \ldots,\ a_1,\ a_0$ from left to right. In other words, the sum in equation (1). Thus, if $a = 2718 = [a_3 a_2 a_1 a_0]$, then $[a_3 a_2] = 27$. We will often use the fact that $[a_k a_{k-1} \ldots a_1 a_0] = 10^n [a_k a_{k-1} \ldots a_n] + [a_{n-1} \ldots a_0]$.

3.1 Examine the Ending Digits

It is exceedingly easy to test if a number a is divisible by 2; simply see if the last digit of a is divisible by 2. The same test works when determining divisibility by 5 or 10. As another example, it turns out that if you want to test divisibility of a by 8, you only need to check if the last three digits of a are divisible by 8.

Ending Digits Trick: Suppose that d divides 10^n for some n. Then d divides a number a if and only if d divides the last n digits of a.

The following table shows all the numbers d from 2 to 102 that divide a power of 10, and the number of ending digits one must check to determine divisibility by d.

To test divisibility by ...	the number of ending digits to examine
2, 5, 10	1
4, 20, 25, 50, 100	2
8, 40	3
16, 80	4
32	5
64	6

Why it works: Suppose that 10^n is divisible by d, or in other words, $10^n \equiv 0 \pmod{d}$. Let a be a number with k digits, and assume that $k \geq n$.

$$\begin{aligned} a &= [a_k a_{k-1} \ldots a_1 a_0] \\ &= 10^n [a_k a_{k-1} \ldots a_n] + [a_{n-1} \ldots a_0] \\ &\equiv [a_{n-1} \ldots a_0] \pmod{d}. \end{aligned}$$

Consequently, d divides a if and only if d divides the last n digits of a, namely, $[a_{n-1} \ldots a_0]$.

Running total: We now have divisibility tests for 2, 4, 5, 8, 10, 16, 20, 25, 32, 40, 50, 64, 80, 100.

3.2 Take a Sum of the Digits

It is well known that a number a is divisible by 3 or 9 if and only if the sum of the digits of a is divisible by 3 or 9, respectively. More generally, we can test divisibility by some numbers by adding together *blocks* of digits, starting from the right. For example, to test divisibility of a by 33, we add the digits of a in blocks of 2. Using this rule, we see that 5210832 is divisible by 33 since $32 + 08 + 21 + 5 = 66$ is clearly divisible by 33.

Sum of Digits Trick: Let d be given, and suppose that $10^n \equiv 1 \pmod{d}$ for some n. Add the digits of a in blocks of n starting from the right, and call the result s. Now a and s leave the same remainder upon division by d; in particular, a is divisible by d if and only if s is divisible by d.

Below are the values of d ($2 \leq d \leq 102$) for which the trick works and the block size is fairly small (at most 4).

d	3	9	11	27	33	37	99	101
block size to add	1	1	2	3	2	3	2	4

Why it works: Suppose that $10^n \equiv 1 \pmod{d}$, and we wish to see if d divides the number $a = [a_k \dots a_0]$. Assuming $k \geq n$, we get

$$\begin{aligned} a = [a_k \dots a_0] &= 10^n [a_k \dots a_n] + [a_{n-1} \dots a_0] \\ &\equiv [a_k \dots a_n] + [a_{n-1} \dots a_0] \pmod{d} \end{aligned}$$

Now letting t be the greatest integer such that $k \geq tn$, and repeating this process on the leftmost term we find

$$\begin{aligned} a &= [a_k \dots a_0] \\ &\equiv [a_k \dots a_n] + [a_{n-1} \dots a_0] \\ &\equiv [a_k \dots a_{2n}] + [a_{2n-1} \dots a_n] + [a_{n-1} \dots a_0] \\ &\vdots \\ &\equiv [a_k \dots a_{tn}] + [a_{tn-1} \dots a_{(t-1)n}] + \cdots + [a_{2n-1} \dots a_n] + [a_{n-1} \dots a_0] \\ &\quad \pmod{d}. \end{aligned}$$

The last expression above is exactly what it means to add the digits of a together in blocks of length n, starting from the right.

Running total: We now have divisibility tests for 2, 3, 4, 5, 8, 9, 10, 11, 16, 20, 27, 25, 32, 33, 37, 40, 50, 64, 80, 99, 100, 101.

3.3 Take an Alternating Sum of Digits

To see if a is divisible by 11, alternately add and subtract the digits of a starting from the right; this alternating sum and a leave the same remainder when divided by 11. As in the previous section, we can extend this idea to blocks of digits. For instance, a is divisible by 91 if and only if the alternating sum of blocks of 3 digits is divisible by 91. To see if 23210481381 is divisible by 91 we consider $381 - 481 + 210 - 23 = 87$. Clearly 87 is not divisible by 91, so neither is 23210481381.

Alternating Sum of Digits Trick: Let d be given, and suppose that $10^n \equiv -1 \pmod{d}$ for some n. Alternately add and subtract the digits of a in blocks of n starting from the right, and call the result s. Now a and s leave the same remainder upon division by d; in particular, a is divisible by d if and only if s is divisible by d.

Below are the values of d ($2 \leq d \leq 102$) for which the alternating sum blocks are at most 4.

d	7	11	13	73	77	91	101
block size to add alternately	3	1	3	4	3	3	2

Why it works: Suppose that $10^n \equiv -1 \pmod{d}$, and we are given the number $a = [a_k \dots a_0]$. Assuming $k \geq n$, we get

$$\begin{aligned} a = [a_k \dots a_0] &= 10^n [a_k \dots a_n] + [a_{n-1} \dots a_0] \\ &\equiv -[a_k \dots a_n] + [a_{n-1} \dots a_0] \pmod{d} \end{aligned}$$

Now letting t be the greatest integer such that $k \geq tn$, and repeating this process on the leftmost term we find

$$\begin{aligned} a &= [a_k \dots a_0] \\ &\equiv -[a_k \dots a_n] + [a_{n-1} \dots a_0] \\ &\equiv -(-[a_k \dots a_{2n}] + [a_{2n-1} \dots a_n]) + [a_{n-1} \dots a_0] \\ &\quad = [a_k \dots a_{2n}] - [a_{2n-1} \dots a_n] + [a_{n-1} \dots a_0] \\ &\vdots \\ &\equiv (-1)^t [a_k \dots a_{tn}] + (-1)^{t-1} [a_{tn-1} \dots a_{(t-1)n}] + \cdots - [a_{2n-1} \dots a_n] + [a_{n-1} \dots a_0] \\ &\quad \pmod{d}. \end{aligned}$$

Running total: We now have divisibility tests for 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 16, 20, 27, 25, 32, 33, 37, 40, 50, 64, 73, 77, 80, 91, 99, 100, 101.

3.4 Trim from the Right

A basic result from elementary number theory tells us that if the greatest common divisor of d and 10 is 1, then there exists a number u such that $10u \equiv 1 \pmod{d}$. Such a number u is called an *inverse* of 10 modulo d and we write $u \equiv 10^{-1} \pmod{d}$.

For instance, $10(4) = 40 \equiv 1 \pmod{13}$ so $4 \equiv 10^{-1}$, mod 13. In fact, any number congruent to 4 mod 13 (e.g. -9), is also an inverse of 10, mod 13.

However, note that 10 has no inverse modulo 25, that is, there is no number u such that $10u \equiv 1 \pmod{25}$.

Knowing the inverse of 10 mod d (if it exists) leads to a nice divisibility test. To test if 283757 is divisible by 13 we can trim off the rightmost digit (7), multiply it by 10^{-1} mod 13 (for example, 4) and add that result to the remaining digits ($28375 + 28 = 28403$). The original number, 283757, is divisible by 13 if and only if the new number, 28403, is divisible by 13. If it is still unclear whether or not the new numbers is divisible by 13, we can repeat the process. Any inverse of 10 will work; instead of 4, we can use -9. Thus 283757 is divisible by 13 if and only if $28375 + (-9)7 = 28312$ is divisible by 13.

$$
\begin{array}{lr}
 & 28375\not{7} \\
- & 63 \\
\hline
 & 2831\not{2} \\
- & 18 \\
\hline
 & 281\not{9} \\
- & 81 \\
\hline
 & 20\not{0} \\
- & 0 \\
\hline
 & 20
\end{array}
$$

Since 20 is not divisible by 13, we conclude that 283757 is not divisible by 13.

Trim from the Right Trick:

- Let $u \equiv 10^{-1} \pmod{d}$, write $a = [a_k \dots a_0]$, and let $a' = [a_k \dots a_1] + u[a_0]$. Then a is divisible by d if and only if a' is divisible by d.

- Let $v \equiv 100^{-1} \pmod{d}$, write $a = [a_k \dots a_0]$, and let $a'' = [a_k \dots a_2] + v[a_1 a_0]$. Then a is divisible by d if and only if a'' is divisible by d.

We list below all the divisors d ($2 \le d \le 102$) where 10 and 100 have an inverse modulo d. An inverse is included in the table if it is a "suitably convenient" number to use in this trimming trick. For instance, $10^{-1} \equiv 61 \pmod{87}$, but 61 is not a particularly easy number to multiply by in mental calculation, so it is omitted. On the other hand, $100^{-1} \equiv -20 \pmod{87}$, and -20 is easy to use, so it is included. Interestingly, the only values for d where neither 10^{-1} nor 100^{-1} is convenient are $d = 63$, 73, and 97.

d	10^{-1}	100^{-1}	d	10^{-1}	100^{-1}	d	10^{-1}	100^{-1}
3	1, −2	1, −2	39	4		73		
7	5, −2	4, −3	41	−4		77		−10
9	1	1	43	−30	40, −3	79	8	
11	−1	1	47		8	81	−8	
13	4, −9	3, −10	49	5		83	25	
17	−5	8, −9	51	−5		87		−20
19	2	4	53		−9	89	9, −80	−8
21	−2	4	57	40	4	91	−9	−10
23	7	3, −20	59	6		93		40
27	−8	10	61	−6		97		
29	3	−20	63			99	10	1
31	−3		67	−20	−2	101	−10	−1
33	10		69	7	−20			
37	−11	10	71	−7				

Why it works: We prove the case for $u \equiv 10^{-1} \pmod{d}$ and simply remark that the proof for $v \equiv 100^{-1} \pmod{d}$ follows similar lines.

Now

$$\begin{aligned} & [a_k \dots a_0] \equiv 0 \pmod{d} \\ \iff & 10[a_k \dots a_1] + [a_0] \equiv 0 \pmod{d} \\ \iff & 10u[a_k \dots a_1] + u[a_0] \equiv 0 \pmod{d} \\ \iff & [a_k \dots a_1] + u[a_0] \equiv 0 \pmod{d} \end{aligned}$$

Running total: We now have divisibility tests for 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 16, 17, 19, 20, 21, 23, 25, 27, 29, 31, 32, 33, 37, 39, 40, 41, 43, 47, 49, 50, 51, 53, 57, 59, 61, 64, 67, 69, 71, 73, 77, 79, 80, 81, 83, 87, 89, 91, 93, 99, 100, 101.

3.5 Trim from the Left

The principle of this trick is that if $100 \equiv h \pmod{d}$ then $100a + b \equiv ha + b \pmod{d}$. For example, to test divisibility by 97, we note that $100 \equiv 3 \pmod{97}$. To apply the principle and see if 27019 is divisible by 97 we can trim off the leftmost digit (2), multiply it by 3 (6) and add that to the remaining digits (7019), but shifted in to the right by two places:

$$\begin{array}{r} \not{2}7019 \\ +\ \ 6 \\ \hline 7619 \end{array}$$

Thus $27019 \equiv 7619 \pmod{97}$. We can continue the process until we arrive at a number that either clearly is or is not divisible by 97.

$$\begin{array}{r} \not{2}\,7019 \\ +\quad 6 \\ \hline \not{7}619 \\ +\ 21 \\ \hline \not{8}29 \\ +\ 24 \\ \hline 53 \end{array}$$

Thus $27019 \equiv 53 \pmod{97}$, and we see that 97 does not divide 27019.

Trim from the Left Trick: Let d be given, let $h \equiv 100 \pmod{d}$ and write $a = [a_k \dots a_0]$. Let a' be the number that results from computing $a_k h$, and adding that to $[a_{k-1} \dots a_0]$ so that the ones digit of $a_k h$ lines up with a_{k-2}. Then $a \equiv a' \pmod{d}$; in particular, a is divisible by d if and only if a' is divisible by d.

d	$100 \pmod{d}$	d	$100 \pmod{d}$	d	$100 \pmod{d}$
7	2	34	-2	53	-6
13	-4	35	-5	95	5
14	2	48	4	96	4
19	5	49	2	97	3
21	-5	51	-2	98	2
32	4	52	-4	102	-2
33	1				

Why it Works: Let d be given, write $a = [a_k \dots a_0]$, and assume $k \geq 2$. If $100 \equiv h \pmod{d}$, then

$$\begin{aligned} a = [a_k \dots a_0] &= a_k 10^k + [a_{k-1} \dots a_0] \\ &\equiv a_k h 10^{k-2} + [a_{k-1} \dots a_0] \pmod{d}. \end{aligned}$$

The effect of adding $a_k h 10^{k-2}$ to $[a_{k-1} \dots a_0]$ is the same as adding $a_k h$ to $[a_{k-1} \dots a_0]$ so that the ones digit of $a_k h$ lines up with a_{k-2}.

Running total: We now have divisibility tests for 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 16, 17, 19, 20, 21, 23, 25, 27, 29, 31, 32, 33, 34, 35, 37, 39, 40, 41, 43, 47, 48, 49, 50, 51, 52, 53, 57, 59, 61, 64, 67, 69, 71, 73, 77, 79, 80, 81, 83, 87, 89, 91, 93, 95, 97, 98, 99, 100, 101, 102.

3.6 Factor the Divisor

Our final trick is not really a divisibility test itself, but is a way to combine other divisibility tricks. For example, to test if a number is divisible by 6, we can check to see if it is divisible by both 2 and 3. A number is divisible by 56, if and only if it is divisible by both 7 and 8.

If 1 is the greatest common divisor of m and n, we say that m and n are *relatively prime*. Observe that $6 = 2 \cdot 3$ and 2 and 3 are relatively prime; also, $56 = 7 \cdot 8$ where 7 and 8 are relatively prime.

Factor the Divisor Trick: Suppose that $d = mn$ where m and n are relatively prime. Then d divides a number a if and only if m divides a and n divides a.

We list below all the d ($2 \leq d \leq 102$) that can be written as the product of numbers that are (pairwise) relatively prime.

d	factors	d	factors	d	factors	d	factors	d	factors
6	2 3	33	3 11	51	3 17	69	3 23	87	3 29
10	2 5	34	2 17	52	4 13	70	2 5 7	88	8 11
12	4 3	35	5 7	54	2 27	72	8 9	90	2 9 5
14	2 7	36	4 9	55	5 11	74	2 37	91	7 13
15	3 5	38	2 19	56	8 7	75	3 25	92	4 23
18	2 9	39	3 13	57	3 19	76	4 19	93	3 31
20	4 5	40	8 5	58	2 29	77	7 11	94	2 47
21	3 7	42	2 3 7	60	4 3 5	78	2 3 13	95	5 19
22	2 11	44	4 11	62	2 31	80	16 5	96	32 3
24	8 3	45	9 5	63	9 7	82	2 41	98	2 49
26	2 13	46	2 23	65	5 13	84	4 3 7	99	9 11
28	4 7	48	16 3	66	2 3 11	85	5 17	100	4 25
30	2 3 5	50	2 25	68	4 17	86	2 43	102	2 3 17

Why it works: Suppose $d = mn$ where m and n are relatively prime. If d divides a, then clearly m divides a and n divides a. Conversely, suppose that m and n each divide a. Then $a = mr$ for some integer r. But if n divides mr, where m and n are relatively prime, one can consider the prime factorization of both sides and see that n must divide r; that is, $r = nx$ for some integer x. So, $a = m(nx) = (mn)x$ and thus mn divides a.

Running total: We now have divisibility tests for all the numbers from 2 to 102!

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